Lattices, codes, and sphere packings

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- Global sphere packings in $\mathbb{R}^n$:
  - An overview of the problem: sphere packings vs lattice sphere packings
  - How to construct dense lattices in high dimension from codes?

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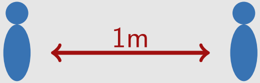
- Global sphere packings in $\mathbb{R}^n$:
    - An overview of the problem: sphere packings vs lattice sphere packings
    - How to construct dense lattices in high dimension from codes?
- Local sphere packings
    - Various problems: Kissing number, spherical codes...
    - How to show the optimality of some configurations in low dimension?

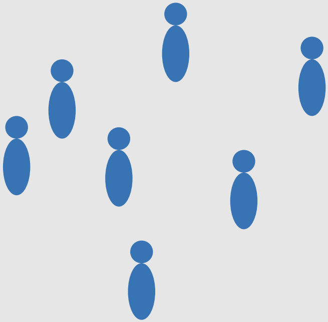
Social distancing and sphere packings

Assume that people should keep one meter distance between themselves...



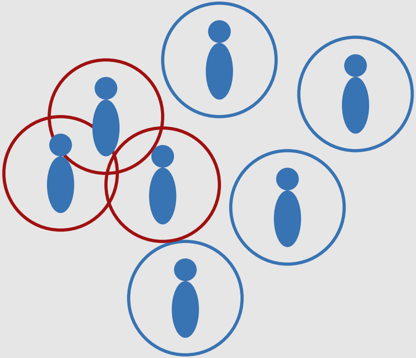
Social distancing and sphere packings

How to deal with a large number of people?



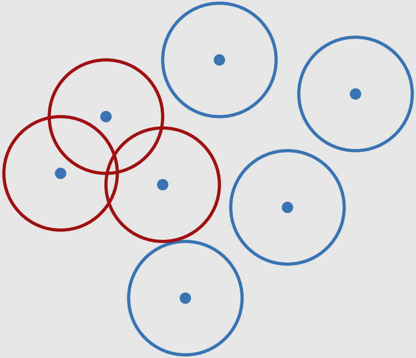
Social distancing and sphere packings

We want non overlapping spheres of radius 0.5m.



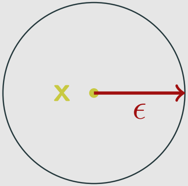
Social distancing and sphere packings

This is the sphere packing problem!



Coding and sphere packings

Consider a noisy channel over $\mathbb{R}^n$: suppose there exists ϵ such that if $x \in \mathbb{R}^n$ is sent, with high probability, the received vector y is in $B(x, \epsilon)$:



Coding and sphere packings

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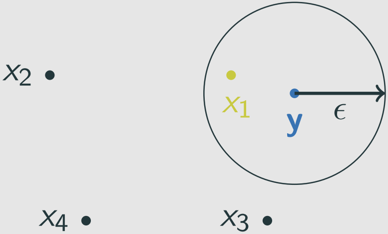
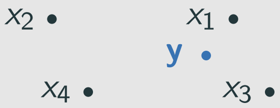
If there is only one codeword in the ball of radius ϵ centred in the received vector y, the receiver can decode the message.



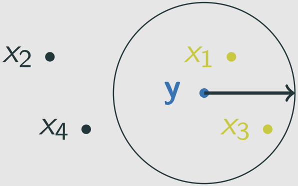
Coding and sphere packings

But if there is more than one word in this ball,



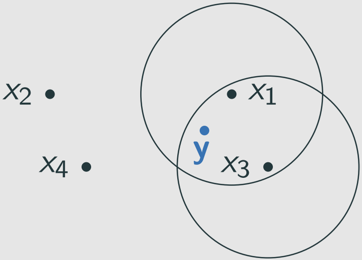
Coding and sphere packings

But if there is more than one word in this ball, the receiver is confused and cannot decode!



Coding and sphere packings

This is equivalent to the fact that the balls of radius ϵ centred in the codewords do not intersect.



Sphere packings

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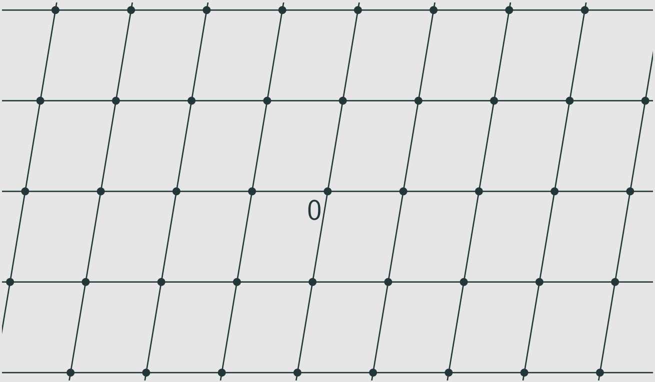
- What if we impose some algebraic structure to the packings, like for linear codes?
- Euclidean lattices provide a way to approach this problem.

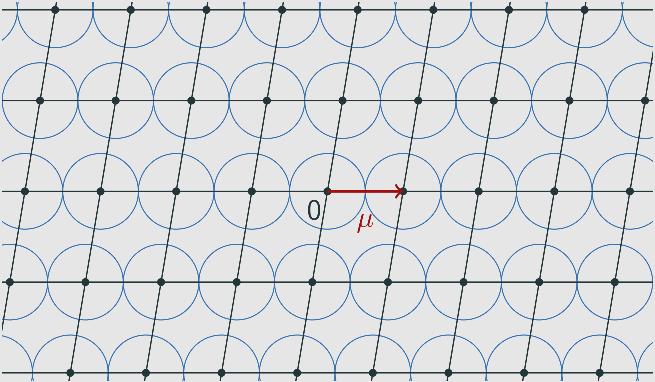
The lattice sphere packing problem

The lattice sphere packing problem consists in finding the biggest proportion of space Δ_n that can be filled by a collection of disjoint spheres having the same radius, with centers at the points of a lattice Λ.



The lattice sphere packing problem

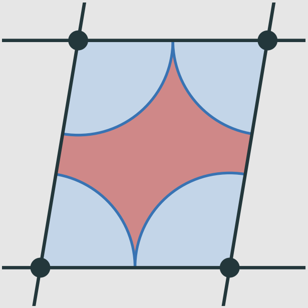
For a given lattice Λ, the best sphere packing associated is given by balls of radius $\mu/2$, where $\mu = \min\{||\lambda||, \lambda \in \Lambda \setminus \{0\}\}$.



The lattice sphere packing problem

The density of this packing is

$$\Delta(\Lambda) = \frac{Vol(B(\mu))}{2^n Vol(\Lambda)}$$



Dimensions 1 and 2

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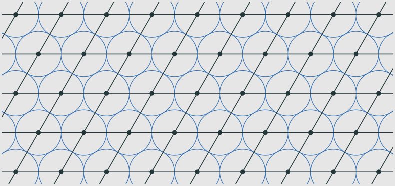
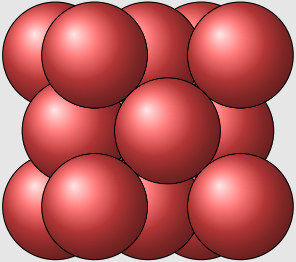
For $n = 2$, the best packing density is $\frac{\pi\sqrt{3}}{6} \approx 0.9069$, and is given by the hexagonal lattice (Lagrange, 1773, best lattice, Thue, 1892 and Fejes Tóth, 1940, best packing).



Dimension 3

For $n = 3$, it is the faced-centered cubic lattice which provides the best density $\frac{\pi\sqrt{2}}{6} \approx 0.74048$ (Kepler conjecture, 1611, Gauss, 1832, best lattice, and Hales, 1998, 2014, best packing).



Solutions for the lattice sphere packing problem

Then we only know the best lattice packings for dimensions $n \leq 8$ and $n = 24$.

Dimension	Lattice	Proved by
4	D_4	Korkine and Zolotareff, 1877
5	D_5	Korkine and Zolotareff, 1877
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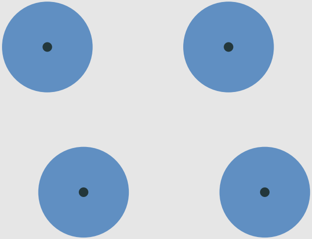
What about high dimensions?

An easy bound for general packings in high dimension

Suppose we have a saturated packing of balls of radius r



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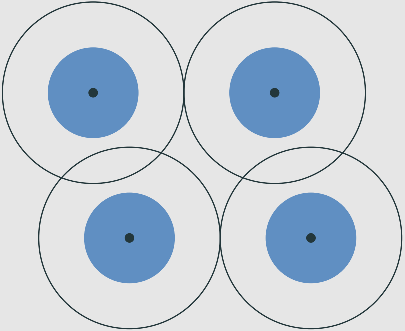
Then, if we double the radius, we cannot have any free point.



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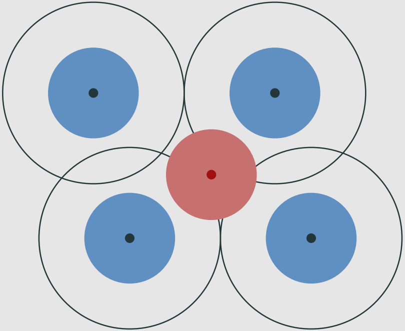

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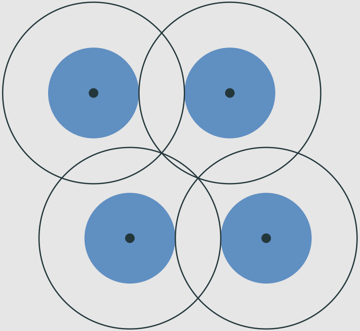
So the balls of radius $2r$ cover the space.



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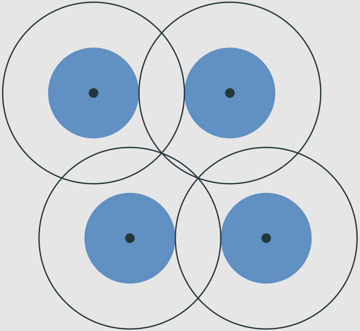
Thus $2^n\Delta \geq 1$, in other words $\Delta \geq \frac{1}{2^n}$.



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However, these results only provide the existence of good lattices, but are not effective.

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Theorem (M., 2017)

For infinitely many dimension n, one can find a lattice $\Lambda \subset \mathbb{R}^n$ satisfying

$$\Delta(\Lambda) > \frac{0.89n \log \log n}{2^n}$$

with $\exp(1.5n \log n(1 + o(1)))$ binary operations.

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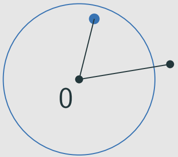

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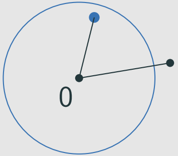

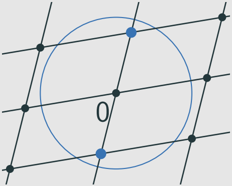

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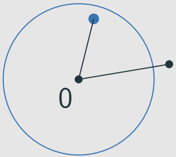

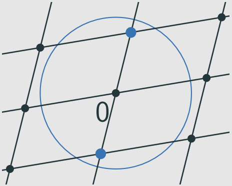

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- So the condition $|B(r) \cap \Lambda \setminus \{0\}| < 2$ is sufficient to conclude $\Delta(\Lambda) \geq \frac{\mathrm{Vol}(B(r))}{2^n \mathrm{Vol}(\Lambda)}$.

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- For $n = 2\ell$ with ℓ prime, Gaborit and Zémor considered finite families of lattices invariant under the action of $\mathbb{Z}/\ell\mathbb{Z}$ via (doubly)-cyclic permutation of coordinates.
- For $n = 2\phi(m)$, Venkatesh constructed infinite families of lattices invariant under the action of mth-roots of unity. Taking $m = \prod_{\substack{q\in\mathbb{P}\\ q\leq X}} q$, he optimized the ratio between m and $2\phi(m)$.

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$$\begin{array}{cccc} \iota & : & K & \to & K_{\mathbb{R}} \\ & & x & \mapsto & (\sigma_1(x), \ldots, \sigma_{r_1}(x), \sigma_{r_1+1}(x), \ldots, \sigma_{r_1+r_2}(x)) \end{array}.$$

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- The map

$$\begin{array}{cccc} \beta & : & K \times K & \to & \mathbb{R} \\ & & (x, y) & \mapsto & \mathrm{tr}(x\bar{y}) \end{array}$$

is a positive-definite symmetric bilinear form, which induces a scalar product $\langle \cdot, \cdot \rangle$ on $K_{\mathbb{R}}$.

(Nordic) Number theory ~~Network~~ Lattices

- Let $K/\mathbb{Q}$ be a number field of degree n.
- Following the real and complex embeddings of $K \to \mathbb{C}$, we can write $n = r_1 + 2r_2$, and there is a natural embedding ι of K into K_R, where $K_R = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^n \simeq K \otimes_{\mathbb{Q}} \mathbb{R}$:

$$\begin{array}{cccc} \iota & : & K & \to & K_{\mathbb{R}} \\ & & x & \mapsto & (\sigma_1(x), \ldots, \sigma_{r_1}(x), \sigma_{r_1+1}(x), \ldots, \sigma_{r_1+r_2}(x)) \end{array}.$$

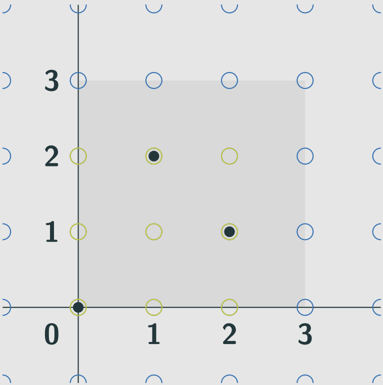
- The map

$$\begin{array}{ccccc} \beta & : & K \times K & \to & \mathbb{R} \\ & & (x, y) & \mapsto & \mathrm{tr}(x\bar{y}) \end{array}$$

  is a positive-definite symmetric bilinear form, which induces a scalar product $\langle \cdot, \cdot \rangle$ on $K_{\mathbb{R}}$.
- The ring of integers $\mathcal{O}_K$, and more generally every fractional ideal $\mathfrak{A}$ of K are free $\mathbb{Z}$-modules of rank n, and thus define lattices in $K_{\mathbb{R}}$.

Lattices from codes: Construction A

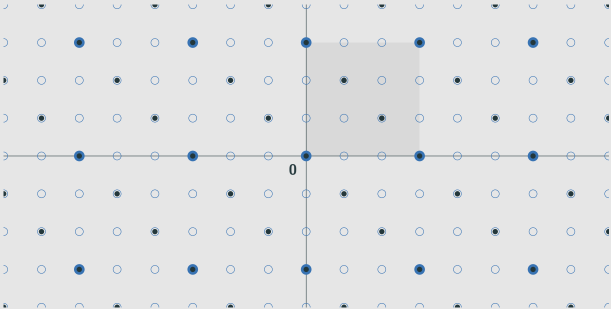
Let p be a prime number, $\pi : \mathbb{Z}^n \to \mathbb{F}_p^n$ the canonical projection, and $C \subset \mathbb{F}_p^n$ a k-dimensional code.



Lattices from codes: Construction A

We define $\Lambda_C = \pi^{-1}(C)$. Then we have $p\mathbb{Z}^n \subset \Lambda_C \subset \mathbb{Z}^n$ and

$$Vol(\Lambda_C) = p^{n-k}$$



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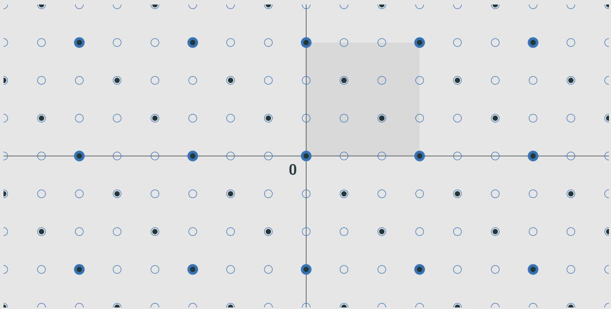

Examples: The famous lattices E_8 and the Leech Lattice Λ_{24} can be obtained via this construction.

Outline of the proof

Theorem (M., 2017)

For infinitely many dimension n, one can find a lattice $\Lambda \subset \mathbb{R}^n$ satisfying

$$\Delta(\Lambda) > \frac{0.89 n \log \log n}{2^n}$$

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- If q is large enough, one gets an analogue of Siegel's mean value theorem.

The kissing number problem

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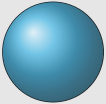

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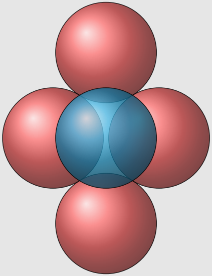

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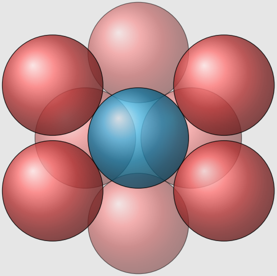

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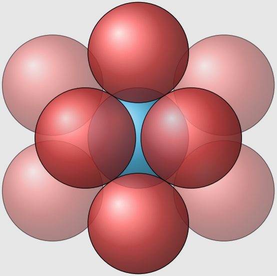

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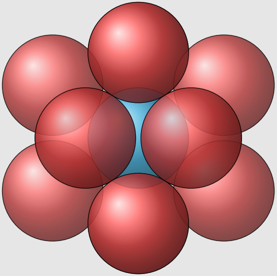

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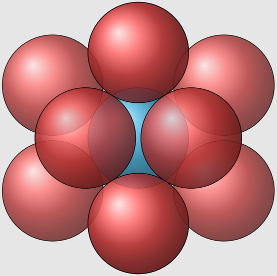

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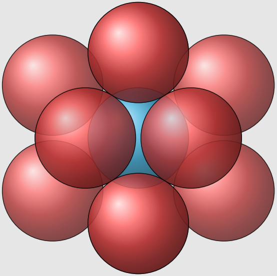

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The lattice kissing number problem: what is the maximal number of shortest vectors achieved by a lattice?

Exponential growth

Let τ_n be the kissing number in dimension n. It is known, from various approaches (Chabauty 1953, Shannon 1959, Wyner 1965) that

$$\frac{\log_2 \tau_n}{n} \geq \log_2 \frac{2}{\sqrt{3}} \simeq 0.2075...$$

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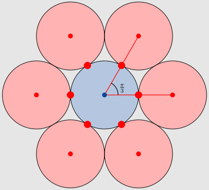
From codes! They come from algebraic geometric codes with exponentially many minimal codewords.



Kissing number:

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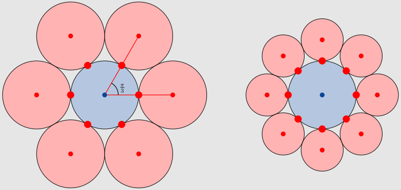
Formulation and generalizations



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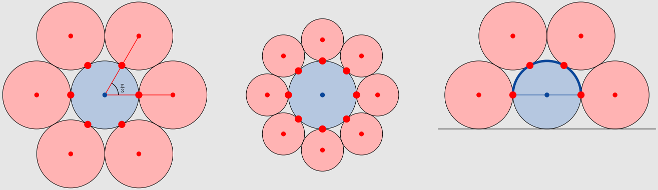

Kissing number of the hemisphere:

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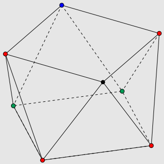
- The square antiprism, the unique optimal θ-spherical code in dimension 3 with $\cos\theta = (2\sqrt{2}-1)/7$ (Schütte-van der Waerden 1951, Danzer 1986).



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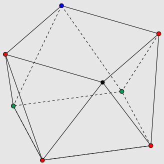

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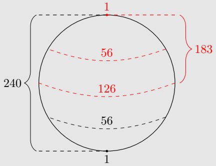

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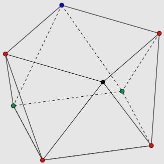

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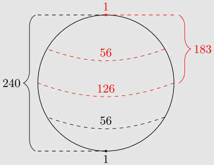

- [Dostert, De Laat, M., 2020]: A general framework to prove optimality and uniqueness of such configurations.

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  These bounds are related to the hierarchies of semidefinite upper bounds used to give upper bounds on the independence number of finite graphs. (Lovász-Schrijver 1991, Lasserre 2001, Laurent 2007)

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- For large problems, SDP solvers only provide approximate solutions in floating point in polynomial time.
- Turning an approximate solution into a rigorous proof is hard!

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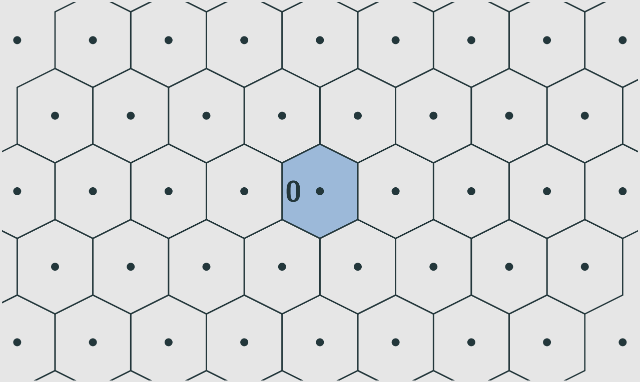
Besides spherical codes, we could apply our method for packing spheres in spheres.

Thank you!

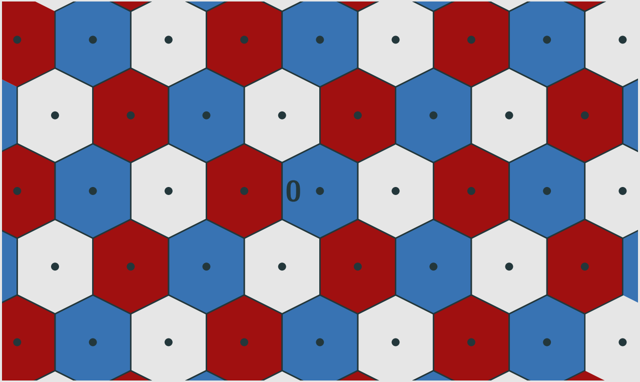
Bonus: Coloring the Voronoi tessellation of a lattice

Recall the Voronoi tesselation of a lattice Λ.



Bonus: Coloring the Voronoi tessellation of a lattice

We want to color this tessellation in such a way that two cells sharing a facet do not receive the same color.



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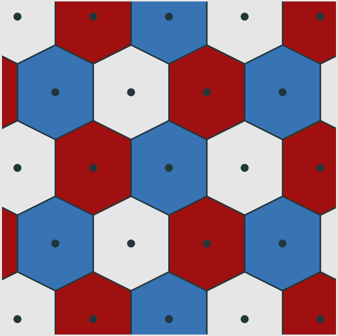
We are colouring a geometric graph G_Λ.



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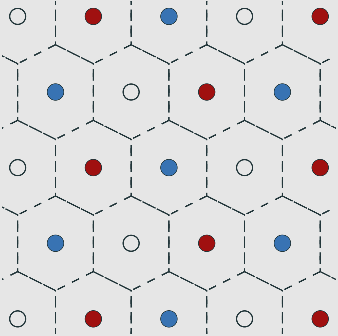

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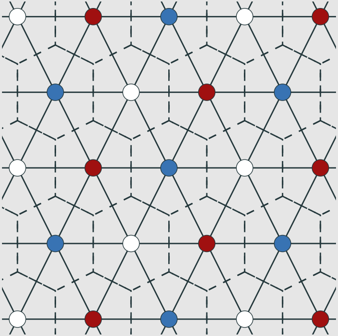

- The vertices: $V = \Lambda$,
- The edges: $\{u, v\} \in E$ if $w = u - v$ is a Voronoi vector of Λ, that is $\mathcal{V}_\Lambda \cap (w + \mathcal{V}_\Lambda)$ is an $(n-1)$-dimensional facet of $\mathcal{V}_\Lambda$.

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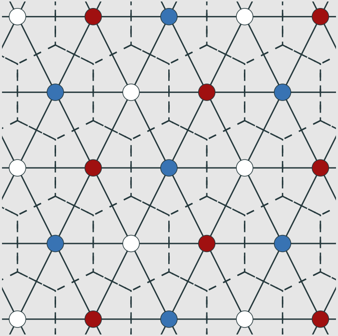

What is the chromatic number $\chi(\Lambda)$ of G_Λ? [Dutour-Sikirić, Madore, M., Vallentin]

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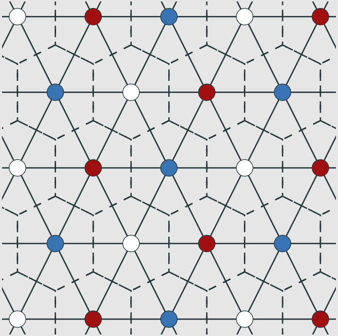

What is the behavior of $\chi(\Lambda)$ with the dimension n ?

- $\chi(\Lambda) \leq 2^n$,
- Expected value: $\chi(\Lambda) \geq 2^{0.099n}$.

What is the chromatic number of the most famous lattices?