Duality and class field theory for curves over p-adic fields

Joint work with Thomas Geisser

29 mai 2020

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so that $W_k \cong \mathbb{Z}$ is the discrete subgroup of G_k generated by Frob.

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Proposition

$$H^i(W_K, \overline{L}^{\times}) \cong \begin{cases} K^{\times} & i = 0 \\ \mathbb{Z} & i = 1 \\ 0 & i > 1 \end{cases}$$

Proof

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Exact sequence of W_k-modules

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We obtain

$$R\Gamma(W_k, \mathcal{O}_L^\times / \mathcal{U}_L^i) \cong \mathcal{O}_K^\times / \mathcal{U}_K^i[0]$$

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which is a symmetric monoidal closed category. In other words, it has a (derived) $\otimes^L$ and (derived) internal $R\mathrm{Hom}$. For example, we have

$$\mathcal{O}_K^\times \otimes^L \mathbb{R} = 0.$$

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Corollary

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$$SK_1(X) := \mathrm{Coker}\left(K_2(K(X)) \xrightarrow{\oplus\delta_x} \bigoplus_{x\in X_0} \kappa(x)^\times\right)$$

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$$\kappa(x)^{\times} \xrightarrow{L.C.F.T} G^{\mathrm{ab}}_{\kappa(x)} = \pi_1^{\mathrm{ab}}(x) \longrightarrow \pi_1^{\mathrm{ab}}(X) \qquad \forall x \in X_0$$

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