PONTRYAGIN DUALITY FOR VARIETIES OVER p-ADIC FIELDS

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Abstract. Under certain assumptions, we define cohomological complexes of locally compact abelian groups associated with varieties over p-adic fields. Then we prove a duality theorem which takes the form of Pontryagin duality between locally compact cohomology groups.

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}_K$ be the ring of integers in K and let $\mathcal{X}$ be a regular, proper and flat scheme over $\mathcal{O}_K$. We denote by $\mathcal{X}_K$ its generic fiber and by $i : \mathcal{X}_s \to \mathcal{X}$ its special fiber. Assuming certain expected property of Bloch's cycle complexes $\mathbb{Z}(n)$, we define an exact triangle

$$R\Gamma_{ar}(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \tag{1}$$

in $\mathbf{D}^b(\mathrm{LCA})$, for any Tate twist $n \in \mathbb{Z}$. Here we denote by LCA be the quasi-abelian category of locally compact abelian groups and by $\mathbf{D}^b(\mathrm{LCA})$ its bounded derived category, as defined in [13]. We also consider the full subcategory FLCA $\subseteq$ LCA consisting of locally compact abelian groups of finite ranks in the sense of [13], and its bounded derived category $\mathbf{D}^b(\mathrm{FLCA})$, which is a closed symmetric monoidal category. We denote by $R\underline{\mathrm{Hom}}(-,-)$ the internal Hom and by $\underline{\otimes}^L$ the tensor product in $\mathbf{D}^b(\mathrm{FLCA})$. For any $A \in$ FLCA, we set

$$R\Gamma_{ar}(-, A(n)) := R\Gamma_{ar}(-, \mathbb{Z}(n))\underline{\otimes}^L A$$

where $\underline{\otimes}^L$ denotes the tensor product in $\mathbf{D}^b(\mathrm{FLCA})$.

Suppose that $\mathcal{X}$ is connected and d-dimensional. We give an alternative construction of the triangle (1) for $n = 0, d$ in order to prove Theorem 1.2 below, which requires the following

Hypothesis 1.1. *The reduced scheme $(\mathcal{X}_s)^{\mathrm{red}}$ is a strict normal crossing scheme, and the complex $R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0))$ is a perfect complex of abelian groups, where $\mathbb{Z}^c(0)$ denotes the Bloch cycle complex in its homological notation* [9] *and $R\Gamma_W(\mathcal{X}_s, -)$ denotes Weil-étale cohomology.*

Theorem 1.2. *Suppose that either $d \leq 2$ or that $\mathcal{X}_s$ satisfies Hypothesis 1.1. Then there is a trace map $H^{2d}_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$ and an equivalence*

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(n)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d-n)), \mathbb{R}/\mathbb{Z}[-2d])$$

in $\mathbf{D}^b(\mathrm{FLCA})$, for $n = 0, d$.

Corollary 1.3. *Suppose that $\mathcal{X}_s$ satisfies Hypothesis 1.1 and suppose moreover that $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ has smooth or log-smooth reduction. Then there is perfect pairing of locally compact abelian groups*

$$H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(n)) \times H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z}(d-n)) \to H^{2d}_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$$

for $n = 0, d$ and any $i \in \mathbb{Z}$.

The ad-hoc definitions given in this paper are of preliminary nature. In fact we conjecture the existence of a cohomology theory on the category of separated schemes of finite type over $\mathrm{Spec}(\mathcal{O}_K)$ satisfying the conclusion of Theorem 1.2 for any Tate twist $n \in \mathbb{Z}$. However, we do not expect Corollary 1.3 to be true in general, since the groups $H^i_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$ are not expected to be locally compact in general. Instead, they should be seen as condensed abelian groups (or in the language used in this paper, as objects of the heart $\mathcal{LH}(\mathrm{LCA})$ of the left t-structure on $\mathbf{D}^b(\mathrm{LCA})$). However, we expect an isomorphism of *compact* groups

$$H^i_{ar}(\mathcal{X}, \mathbb{R}/\mathbb{Z}(n)) \simeq H^{2d+1-i}_{ar}(\mathcal{X}_s, Ri^!\mathbb{Z}(d-n))^D$$

for any $i, n \in \mathbb{Z}$, where $(-)^D$ denotes the Pontryagin dual. Concerning the relationship between

$$R\Gamma_{ar}(-, A(n)) := R\Gamma_{ar}(-, \mathbb{Z}(n)) \underline{\otimes}^L A$$

and known cohomology theories, we expect the following

Conjecture 1.4. *We set $X := \mathcal{X}_K$. For any prime l, possibly $l = p$, one has an isomorphism*

$$R\Gamma_{ar}(X, \mathbb{Z}(n)) \widehat{\otimes} \mathbb{Z}_l \simeq R\Gamma_{et}(X, \mathbb{Z}_l(n))$$

where $(-)\widehat{\otimes}\mathbb{Z}_l := \mathrm{holim}(- \otimes^L \mathbb{Z}/l^\bullet)$ is the l-adic completion functor. For $l \neq p$, one has an isomorphism

$$R\Gamma_{ar}(X, \mathbb{Z}_l(n)) \simeq R\Gamma_{et}(X, \mathbb{Z}_l(n))$$

For $l = p$, one has an isomorphism

$$R\Gamma_{ar}(X, \mathbb{Q}_p(n)) \simeq R\Gamma_{syn}(X, n)$$

where the right hand side is Niziol-Nekovar syntomic cohomology, and an isomorphism between

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \widehat{\otimes} \mathbb{Q}_p \simeq R\Gamma_{ar}(\mathcal{X}, \mathbb{Q}_p(n))$$

and Fontaine-Messing syntomic cohomology. Moreover, we have

$$\dim_{\mathbb{Q}_l} H^i_{ar}(X, \mathbb{Q}_l(n)) = \dim_{\mathbb{R}} H^i_{ar}(X, \mathbb{R}(n))$$

for any $i \in \mathbb{Z}$ and any $l \neq p$. In particular, the left hand side is independent on $l \neq p$. Finally, we have

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{R}(n)) \simeq R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}(n)) \otimes \mathbb{R}$$

where the right hand side is motivic Weil-h cohomology [8]*, and an exact triangle*

$$R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(d-n))_{\mathbb{R}} \to R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}(n))_{\mathbb{R}} \to R\Gamma_{ar}(X, \mathbb{R}(n)).$$

Note that Conjecture 1.4 would provide a map

$$R\Gamma_{syn}(X,n)\simeq R\Gamma_{ar}(X,\mathbb{Q}_p(n))\longrightarrow R\Gamma_{ar}(X,\mathbb{Z}(n))\widehat{\otimes}\mathbb{Q}_p\simeq R\Gamma_{et}(X,\mathbb{Q}_p(n)).$$

This map is expected to be an isomorphism if and only if $n\geq d$. The existence of a cohomology theory satisfying these properties and in particular Pontryagin duality was sugested by the "Weil-Arakelov cohomology" of arithmetic schemes, which is conditionally defined in [4] for proper regular schemes over $\mathrm{Spec}(\mathbb{Z})$. One advantage that the theory $R\Gamma_{ar}(-,A(n))$ should have over étale motivic cohomology $R\Gamma_{et}(-,\mathbb{Z}(n))$ for varieties over p-adic fields is the fact that $R\Gamma_{ar}(-,A(n))$ is expected to satisfy a duality as in Conjecture 1.2, whereas étale motivic cohomology does not satisfy any duality. It gives for example a new viewpoint on Class Field Theory, as we shall explain in a subsequent paper.

2. Locally compact abelian groups

2.1. **Derived ∞-categories.** Let $\mathcal{U}$ be a Grothendieck universe and let A be a $\mathcal{U}$-small additive category. Let $\mathrm{C}^b(A)$ be the $\mathcal{U}$-small differential graded category of bounded complexes of objects in A and let $\mathcal{N}\subset\mathrm{C}^b(A)$ be a full subcategory which is closed under the formation of shifts and under the formation of mapping cones. Then $\mathrm{N_{dg}}(\mathrm{C}^b(A))$ is a $\mathcal{U}$-small stable ∞-category and $\mathrm{N_{dg}}(\mathcal{N})$ is a stable ∞-subcategory of $\mathrm{N_{dg}}(\mathrm{C}^b(A))$ [18, Proposition 1.3.2.10], where $\mathrm{N_{dg}}(-)$ denotes the differential graded nerve [18, Construction 1.3.1.6]. The Verdier quotient $\mathrm{N_{dg}}(\mathrm{C}^b(A))/\mathrm{N_{dg}}(\mathcal{N})$ is defined as the cofiber of the functor

$$\mathrm{N_{dg}}(\mathcal{N})\to\mathrm{N_{dg}}(\mathrm{C}^b(A))$$

computed in the ∞-category $\mathrm{Cat}_\infty^{\mathrm{ex}}$ of $\mathcal{U}$-small stable ∞-categories and exact functors. Let S be the set of morphisms f in $\mathrm{N_{dg}}(\mathrm{C}^b(A))$ such that $\mathrm{Cofib}(f)\in\mathrm{N_{dg}}(\mathcal{N})$. Then the functor

$$\mathrm{N_{dg}}(\mathrm{C}^b(A))\to\mathrm{N_{dg}}(\mathrm{C}^b(A))/\mathrm{N_{dg}}(\mathcal{N})$$

induces an equivalence [3, Theorem 1.3]

$$\mathrm{N_{dg}}(\mathrm{C}^b(A))[S^{-1}]\xrightarrow{\sim}\mathrm{N_{dg}}(\mathrm{C}^b(A))/\mathrm{N_{dg}}(\mathcal{N}).$$

Moreover, we have an equivalence of categories

$$h(\mathrm{N_{dg}}(\mathrm{C}^b(A))/\mathrm{N_{dg}}(\mathcal{N}))\simeq h(\mathrm{N_{dg}}(\mathrm{C}^b(A)))/h(\mathrm{N_{dg}}(\mathcal{N})))$$

where $h(-)$ denotes the homotopy category, and the right hand side is the classical Verdier quotient [2, Proof of Prop. 5.9]. Note that the homotopy category of a stable ∞-category is triangulated [18, Theorem 1.1.2.14].

If A is a quasi-abelian category in the sense of [26] (or more generally an exact category), we define its bounded derived ∞-category

$$\mathbf{D}^b(A):=\mathrm{N_{dg}}(\mathrm{C}^b(A))/\mathrm{N_{dg}}(\mathcal{N})\simeq\mathrm{N_{dg}}(\mathrm{C}^b(A))[S^{-1}]$$

where $\mathcal{N}\subset\mathrm{C}^b(A)$ is the full subcategory of strictly acyclic complexes, and S is the set of strict quasi-isomorphisms. The homotopy category

$$\mathrm{D}^b(A):=h(\mathbf{D}^b(A))$$

is equivalent to the bounded derived category of the quasi-abelian category A in the sense of [26].

2.2. **The category $\mathbf{D}^b(\mathrm{LCA})$.** We denote by LCA be the quasi-abelian category of locally compact abelian groups, which are elements of a universe $\mathcal{U}'$. Let $\mathcal{U}$ be a larger universe such that LCA is $\mathcal{U}$-small. Let FLCA $\subset$ LCA be the quasi-abelian category of locally compact abelian groups of finite ranks in the sense of [13, Def. 2.6]. Let $\mathbf{D}^b(\mathrm{LCA})$ (resp. $\mathbf{D}^b(\mathrm{FLCA})$) be the bounded derived ∞-category of LCA (resp. of FLCA). Hence $\mathbf{D}^b(\mathrm{LCA})$ (resp. $\mathbf{D}^b(\mathrm{FLCA})$) is a stable ∞-category in the sense of [18] whose homotopy category is the bounded derived category $\mathrm{D}^b(\mathrm{LCA})$ (resp. $\mathrm{D}^b(\mathrm{FLCA})$) as defined in [13]. It is more convenient to work with the derived ∞-category $\mathbf{D}^b(\mathrm{LCA})$ rather than with its homotopy category. For example, let $\mathrm{Fun}(\Delta^1, \mathbf{D}^b(\mathrm{LCA}))$ be the ∞-category of arrows in $\mathbf{D}^b(\mathrm{LCA})$. Taking the mapping fiber (or cofiber) of a morphism defines a functor (see [18, Remark 1.1.1.7])

$$\begin{array}{rccc} \mathrm{Fib}: & \mathrm{Fun}(\Delta^1, \mathbf{D}^b(\mathrm{LCA})) & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ & C \to C' & \longmapsto & C \times_{C'} 0 \end{array}.$$

The stable ∞-category $\mathbf{D}^b(\mathrm{LCA})$ is endowed with a t-structure by [13], since a t-structure on a stable ∞-category is defined as a t-structure on its homotopy category [18, Definition 1.2.1.4]. This also applies to $\mathbf{D}^b(\mathrm{FLCA})$. Let TA be the quasi-abelian category of topological abelian groups, and define $\mathbf{D}^b(\mathrm{TA})$ and $\mathrm{D}^b(\mathrm{TA})$ as above. The inclusions FLCA $\subset$ LCA $\subset$ TA send strict quasi-isomorphisms to strict quasi-isomorphisms, hence induce functors

$$\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA}) \to \mathbf{D}^b(\mathrm{TA}).$$

The functor disc : TA $\to$ Ab, sending a topological abelian group to its underlying discrete abelian group, sends strict quasi-isomorphisms to usual quasi-isomorphism. This yields a functor

$$\mathrm{disc} : \mathbf{D}^b(\mathrm{TA}) \to \mathbf{D}^b(\mathrm{Ab}).$$

We denote by X^D the Pontryagin dual of the locally compact abelian group X. Recall that $X^D := \underline{\mathrm{Hom}}(X, \mathbb{R}/\mathbb{Z})$ is the group of continuous morphisms $X \to \mathbb{R}/\mathbb{Z}$ endowed with the compact-open topology, and that Pontryagin duality gives an isomorphism of locally compact groups

$$X \xrightarrow{\sim} X^{DD}.$$

Then the functor $(-)^D$ sends strict quasi-isomorphisms to strict quasi-isomorphisms and locally compact compact abelian groups of finite ranks to locally compact groups of finite ranks. We obtain equivalences of ∞-categories

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{LCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ X & \longmapsto & X^D \end{array}$$

and

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{FLCA}) \\ X & \longmapsto & X^D \end{array}.$$

In [13], the authors define functors

$$R\mathrm{Hom}_{\mathrm{LCA}}(-,-) : \mathrm{D}^b(\mathrm{LCA})^{\mathrm{op}} \times \mathrm{D}^b(\mathrm{LCA}) \to \mathrm{D}^b(\mathrm{TA})$$

and

$$R\mathrm{Hom}_{\mathrm{FLCA}}(-,-) : \mathrm{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathrm{D}^b(\mathrm{FLCA}) \to \mathrm{D}^b(\mathrm{FLCA}).$$

The construction of the functor $R\mathrm{Hom}_{\mathrm{FLCA}}(-,-)$ actually gives a functor of stable ∞-categories

$$R\underline{\mathrm{Hom}}(-,-) : \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{FLCA}).$$

Indeed, let I (resp. P) be the additive category of divisible (resp. codivisible) locally compact abelian groups I (resp. P) of finite ranks such that $I_{\mathbb{Z}} = 0$ (such that $P_{\mathbb{S}^1} = 0$), see [13, Def. 3.2]. Define

$$\mathbf{D}^b(\mathrm{I}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{I}))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}_{\mathrm{I}})$$

where $\mathcal{N}_{\mathrm{I}} \subset \mathrm{C}^b(\mathrm{I})$ is the dg-subcategory of strictly acyclic bounded complexes. We define similarly

$$\mathbf{D}^b(\mathrm{P}) := \mathrm{N}_{\mathrm{dg}}(\mathrm{C}^b(\mathrm{P}))/\mathrm{N}_{\mathrm{dg}}(\mathcal{N}_{\mathrm{P}}).$$

The functor

$$\mathbf{D}^b(\mathrm{I}) \to \mathbf{D}^b(\mathrm{FLCA}) \tag{2}$$

is an exact functor of stable ∞-categories which induces an equivalences between the corresponding homotopy categories by [13, Cor. 3.10]. It follows that (2) is an equivalence of stable ∞-categories. Similarly $\mathbf{D}^b(\mathrm{P}) \to \mathbf{D}^b(\mathrm{FLCA})$ is an equivalence. We may therefore define

$$R\underline{\mathrm{Hom}}(-,-) : \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathbf{D}^b(\mathrm{FLCA}) \xleftarrow{\sim} \mathbf{D}^b(\mathrm{P})^{\mathrm{op}} \times \mathbf{D}^b(\mathrm{I}) \to \mathbf{D}^b(\mathrm{FLCA})$$

since the functor

$$\begin{array}{ccc} \mathrm{C}^b(\mathrm{P})^{\mathrm{op}} \times \mathrm{C}^b(\mathrm{I}) & \longrightarrow & \mathrm{C}^b(\mathrm{FLCA}) \\ (P,I) & \longmapsto & \underline{\mathrm{Hom}}^{\bullet}(P,I) := \mathrm{Tot}(\underline{\mathrm{Hom}}(P,I)) \end{array}$$

sends a pair of strict quasi-isomorphisms to a strict quasi-isomorphism [13, Cor. 3.7]. Here $\underline{\mathrm{Hom}}(P,I)$ is the double complex of continuous maps endowed with the compact-open topology, and Tot denotes the total complex. Note that the Pontryagin dual X^D is given by the functor

$$\begin{array}{cccc} R\underline{\mathrm{Hom}}(-,\mathbb{R}/\mathbb{Z}) : & \mathbf{D}^b(\mathrm{FLCA})^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b(\mathrm{FLCA}) \\ & X & \longmapsto & X^D \end{array}.$$

Following [13], we define the derived topological tensor product

$$\begin{array}{ccc} \mathbf{D}^b(\mathrm{FLCA}) \times \mathbf{D}^b(\mathrm{FLCA}) & \longrightarrow & \mathbf{D}^b(\mathrm{FLCA}) \\ (X,Y) & \longmapsto & X \underline{\otimes}^L Y := R\underline{\mathrm{Hom}}(X, Y^D)^D \end{array}.$$

Lemma 2.1. *The functor* $\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA})$ *is an exact and fully faithful functor of stable ∞-categories.*

Proof. We first show that this functor is exact. It sends zero objects to zero objects, so it is enough to check that it sends cofiber sequences to cofiber sequences. Let

$$X \xrightarrow{f} Y \to Z \tag{3}$$

be a cofiber sequence in $\mathbf{D}^b(\mathrm{FLCA})$. The map f is equivalent to a morphism of complexes $f_0 : P \to I$, where $P \in \mathrm{C}^b(\mathrm{P})$ (resp. $I \in \mathrm{C}^b(\mathrm{I})$) is given with a strict quasi-isomorphism $P \xrightarrow{\sim} X$ (resp. $Y \xrightarrow{\sim} I$). Then $\mathrm{Cone}(f_0) \in \mathbf{D}^b(\mathrm{FLCA})$ and

$$P \xrightarrow{f_0} I \to \mathrm{Cone}(f_0) \tag{4}$$

is a cofiber sequence in $\mathbf{D}^b(\mathrm{FLCA})$ equivalent to (3). Similarly, (4) is a cofiber sequence equivalent to (3) in $\mathbf{D}^b(\mathrm{LCA})$. Hence (3) is a cofiber sequence in $\mathbf{D}^b(\mathrm{LCA})$.

Now we show that the functor is fully faithful. The functors $R\mathrm{Hom}_{\mathrm{LCA}}(-,-)$ and $R\mathrm{Hom}_{\mathrm{FLCA}}(-,-)$ induce the same functor

$$\mathrm{D}^b(\mathrm{FLCA})^{\mathrm{op}} \times \mathrm{D}^b(\mathrm{FLCA}) \to \mathrm{D}^b(\mathrm{TA}).$$

Moreover, for any $X, Y \in \mathrm{D}^b(\mathrm{FLCA})$ we have

$$\mathrm{disc}(H^0(R\mathrm{Hom}_{\mathrm{LCA}}(X,Y))) \simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{LCA})}(X,Y)$$

and

$$\mathrm{disc}(H^0(R\mathrm{Hom}_{\mathrm{FLCA}}(X,Y))) \simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{FLCA})}(X,Y).$$

Therefore, the map

$$\mathrm{Hom}_{\mathrm{D}^b(\mathrm{FLCA})}(X,Y) \to \mathrm{Hom}_{\mathrm{D}^b(\mathrm{LCA})}(X,Y)$$

is an isomorphism of abelian groups, i.e. $\mathrm{D}^b(\mathrm{FLCA}) \to \mathrm{D}^b(\mathrm{LCA})$ is fully faithful. Hence

$$\mathbf{D}^b(\mathrm{FLCA}) \to \mathbf{D}^b(\mathrm{LCA}) \tag{5}$$

is an exact functor of stable ∞-categories which induces a fully faithful functor between the corresponding homotopy categories. It follows that (5) is fully faithful.

□

Therefore we may identify $\mathbf{D}^b(\mathrm{FLCA})$ with its essential image in $\mathbf{D}^b(\mathrm{LCA})$. Recall also that $\mathbf{D}^b(\mathrm{LCA})$ has a t-structure. We denote its heart by $\mathcal{LH}(\mathrm{LCA})$. It is an abelian category containing LCA as a full subcategory.

Lemma 2.2. *Let $X \in \mathbf{D}^b(\mathrm{LCA})$. Then $X \in \mathbf{D}^b(\mathrm{FLCA})$ if and only if $H^i(X) \in \mathcal{LH}(\mathrm{FLCA})$ for any i.*

Proof. If $X \to Y \to Z$ is a fiber sequence in $\mathbf{D}^b(\mathrm{LCA})$ such that $X, Z \in \mathbf{D}^b(\mathrm{FLCA})$, then $Y \in \mathbf{D}^b(\mathrm{FLCA})$. Indeed, Y is then equivalent to the fiber of a morphism $Z \to X[1]$ in $\mathbf{D}^b(\mathrm{FLCA})$, hence Y belongs to $\mathbf{D}^b(\mathrm{FLCA})$ by the previous lemma.

Let $X \in \mathbf{D}^b(\mathrm{LCA})$ such that $H^i(X) \in \mathcal{LH}(\mathrm{FLCA})$ for any i. Suppose that X is cohomologicaly concentrated in degrees $\leq n$, and consider the fiber sequence

$$\tau^{<n} X \to X \to H^n(X)[-n].$$

Since $H^n(X)[-n] \in \mathbf{D}^b(\mathrm{FLCA})$, we have $X \in \mathbf{D}^b(\mathrm{FLCA})$ if and only if $\tau^{<n}X$. We obtain $X \in \mathbf{D}^b(\mathrm{FLCA})$ by induction on n since X is bounded and since any zero object belongs to $\mathbf{D}^b(\mathrm{FLCA})$.

The converse is obvious.

□

The inclusion $\mathrm{Ab} \subset \mathrm{LCA}$ induces an exact functor

$$i : \mathbf{D}^b(\mathrm{Ab}) \to \mathbf{D}^b(\mathrm{LCA})$$

Proposition 2.3. *The exact functor* $i : \mathbf{D}^b(\mathrm{Ab}) \to \mathbf{D}^b(\mathrm{LCA})$ *is fully faithful and left adjoint to*
$$\mathrm{disc} : \mathbf{D}^b(\mathrm{LCA}) \to \mathbf{D}^b(\mathrm{Ab}).$$

Proof. The functor
$$\mathrm{C}^b(\mathrm{Ab}) \xrightarrow{i} \mathrm{C}^b(\mathrm{LCA}) \xrightarrow{\mathrm{disc}} \mathrm{C}^b(\mathrm{Ab})$$
is isomorphic to the identity functor of $\mathrm{C}^b(\mathrm{Ab})$. We obtain a natural transformation
$$\mathrm{Id}_{\mathbf{D}^b(\mathrm{Ab})} \xrightarrow{\sim} \mathrm{disc} \circ i. \tag{6}$$
Similarly, there is a natural transformation
$$i \circ \mathrm{disc} \to \mathrm{Id}_{\mathbf{D}^b(\mathrm{Ab})}.$$

Let $X \in \mathbf{D}^b(\mathrm{Ab})$ and let $Y \in \mathbf{D}^b(\mathrm{LCA})$. Let $F \xrightarrow{\sim} X$ be a bounded flat resolution, and let $Y \xrightarrow{\sim} D$ be a strict quasi-isomorphism where D is a bounded complex of divisible locally compact abelian groups. Then F is a bounded complex of codivisible discrete groups F^i (in particular, such that $F^i_{\mathbb{S}^1} = 0$). Therefore, we have
$$R\mathrm{Hom}_{\mathrm{LCA}}(i(X), Y) \simeq \underline{\mathrm{Hom}}^\bullet(F, D) := \mathrm{Tot}(\underline{\mathrm{Hom}}(F, D))$$
by [13, Corollary 4.7], where $\underline{\mathrm{Hom}}(F, D)$ is the double complex of continuous maps endowed with the compact-open topology, and Tot denotes the total complex. We obtain
$$\begin{aligned} \mathrm{disc}\,(R\mathrm{Hom}_{\mathrm{LCA}}(i(X), Y)) &\simeq \mathrm{disc}\,(\underline{\mathrm{Hom}}^\bullet(F, D)) \\ &\simeq \mathrm{Hom}^\bullet(F, \mathrm{disc}(D)) \\ &\simeq R\mathrm{Hom}(X, \mathrm{disc}(Y)). \end{aligned}$$
In view of [13, Proposition 4.12] we have
$$\begin{aligned} H^0(\mathrm{disc}\,(R\mathrm{Hom}_{\mathrm{LCA}}(i(X), Y[-n]))) &\simeq \mathrm{disc}(H^0\,(R\mathrm{Hom}_{\mathrm{LCA}}(i(X), Y[-n]))) \\ &\simeq \mathrm{Hom}_{\mathrm{D}^b(\mathrm{LCA})}(i(X), Y[-n]) \\ &\simeq \pi_0(\mathrm{Map}_{\mathbf{D}^b(\mathrm{LCA})}(i(X), \Omega^n Y)) \\ &\simeq \pi_n(\mathrm{Map}_{\mathbf{D}^b(\mathrm{LCA})}(i(X), Y)) \end{aligned}$$
where $\Omega(-) := 0 \times_{(-)} 0$ is the loop space functor. Similarly we have
$$H^0(R\mathrm{Hom}(X, \mathrm{disc}(Y[-n]))) \simeq \pi_n(\mathrm{Map}_{\mathbf{D}^b(\mathrm{Ab})}(X, \mathrm{disc}(Y))).$$
Hence the map
$$\mathrm{Map}_{\mathbf{D}^b(\mathrm{LCA})}(i(X), Y) \to \mathrm{Map}_{\mathbf{D}^b(\mathrm{Ab})}(X, \mathrm{disc}(Y))$$
is an equivalence of ∞-groupoids. The result then follows from [17, Proposition 5.2.2.8] and from the fact that the unit transformation (6) is an equivalence. □

Definition 2.4. *An object* $X \in \mathbf{D}^b(\mathrm{LCA})$ *lies in the essential image of the functor* $i : \mathbf{D}^b(\mathrm{Ab}) \to \mathbf{D}^b(\mathrm{LCA})$ *if and only if the co-unit map* $i \circ \mathrm{disc}(X) \to X$ *is an equivalence. Such an object* $X \in \mathbf{D}^b(\mathrm{LCA})$ *is called* discrete.

Notation 2.5. *If* $X, Y \in \mathbf{D}^b(\mathrm{Ab})$*, then we denote by* $R\mathrm{Hom}(X, Y) \in \mathbf{D}^b(\mathrm{Ab}) \subseteq \mathbf{D}^b(\mathrm{LCA})$ *the usual* $R\mathrm{Hom}$ *seen as an object of* $\mathbf{D}^b(\mathrm{LCA})$.

Lemma 2.6. *Let $X, Y \in \mathbf{D}^b(\mathrm{Ab})$. If the image of X and Y belong to $\mathbf{D}^b(\mathrm{FLCA})$, then there is a canonical map*

$$i(R\mathrm{Hom}(X,Y)) \to R\underline{\mathrm{Hom}}(X,Y).$$

Moreover, if X, Y are perfect complexes of abelian groups, then this map is an equivalence.

Proof. Let $P \xrightarrow{\sim} X$ and $Y \xrightarrow{\sim} I$ be strict quasi-isomorphisms where $P \in \mathrm{C}^b(\mathrm{P})$ (resp. $I \in \mathrm{C}^b(\mathrm{I})$). Let P^δ and I^δ the underlying complexes of discrete abelian groups. Then the maps $P^\delta \xrightarrow{\sim} X$ and $Y \xrightarrow{\sim} I^\delta$ are quasi-isomorphisms in the usual sense. Hence we have $\mathrm{Hom}^\bullet(X, I^\delta) \simeq R\mathrm{Hom}(X,Y)$, where $\mathrm{Hom}^\bullet$ denotes the total complex of the double complex of morphisms of abelian groups. We denote by $\underline{\mathrm{Hom}}^\bullet(P, I)$ the total complex of the double complex of continuous morphisms endowed with the compact-open topology. Then we have morphisms

$$R\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}^\bullet(X, I^\delta) \to \underline{\mathrm{Hom}}^\bullet(X, I) \to \underline{\mathrm{Hom}}^\bullet(P, I) \simeq R\underline{\mathrm{Hom}}(X,Y).$$

Suppose now that X and Y are perfect complexes of abelian groups. We may suppose that X^i is a finitely generated free abelian group for all $i \in \mathbb{Z}$, zero for almost all i, and similarly for Y. We have a strict quasi-isomorphism

$$Y \xrightarrow{\sim} I := \mathrm{Tot}[Y \otimes \mathbb{R} \to Y \otimes \mathbb{R}/\mathbb{Z}]$$

where $[Y \otimes \mathbb{R} \to Y \otimes \mathbb{R}/\mathbb{Z}]$ is seen as a double complex and Tot is the total complex. Then $X \in \mathrm{C}^b(\mathrm{P})$ and $I \in \mathrm{C}^b(\mathrm{I})$ and we have a strict quasi-isomorphism

$$\mathrm{Hom}^\bullet(X,Y) \xrightarrow{\sim} \underline{\mathrm{Hom}}^\bullet(X, I).$$

We obtain

$$R\mathrm{Hom}(X,Y) \simeq \mathrm{Hom}^\bullet(X,Y) \xrightarrow{\sim} \underline{\mathrm{Hom}}^\bullet(X, I) \simeq R\underline{\mathrm{Hom}}(X,Y).$$

□

2.3. **Profinite completion.**

Definition 2.7. *We define a functor*

$$\begin{array}{rccc} (-)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}: & \mathbf{D}^b(\mathrm{Ab}) & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ & X & \longmapsto & (i(\mathrm{colim}\, R\mathrm{Hom}(X, \mathbb{Z}/m)))^D \end{array},$$

where we compute $R\mathrm{Hom}(X, \mathbb{Z}/m)$ and the colimit $\mathrm{colim}\, R\mathrm{Hom}(X, \mathbb{Z}/m)$ over m in the ∞-category $\mathbf{D}^b(\mathrm{Ab})$. We define similarly

$$\begin{array}{rccc} (-)\widehat{\underline{\otimes}}\mathbb{Z}_p: & \mathbf{D}^b(\mathrm{Ab}) & \longrightarrow & \mathbf{D}^b(\mathrm{LCA}) \\ & X & \longmapsto & (i(\mathrm{colim}\, R\mathrm{Hom}(X, \mathbb{Z}/p^\bullet)))^D \end{array}.$$

For any $X \in \mathbf{D}^b(\mathrm{LCA})$ we define

$$R\underline{\mathrm{Hom}}(X, \mathbb{Z}/m) := \mathrm{Fib}(X^D \xrightarrow{m} X^D)$$

and

$$X \underline{\otimes}^L \mathbb{Z}/m := \mathrm{Cofib}(X \xrightarrow{m} X).$$

Proposition 2.8. *Let $X \in \mathbf{D}^b(\mathrm{Ab})$. Suppose that $R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m) \in \mathbf{D}^b(\mathrm{LCA})$ is discrete for any m. Then we have an equivalence*

$$X \widehat{\underline{\otimes}} \widehat{\mathbb{Z}} \simeq \varprojlim \left(i(X) \underline{\otimes}^L \mathbb{Z}/m \right)$$

where the limit is computed in the ∞-category $\mathbf{D}^b(\mathrm{LCA})$ and an equivalence

$$\mathrm{disc}(X \widehat{\underline{\otimes}} \widehat{\mathbb{Z}}) \simeq X \widehat{\otimes} \widehat{\mathbb{Z}} := \varprojlim (X \otimes^L \mathbb{Z}/m) \in \mathbf{D}^b(\mathrm{Ab}).$$

Proof. The co-unit map

$$i \circ \mathrm{disc}\, R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m) \to R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m)$$

is an equivalence by assumption. Moreover we have

$$R\mathrm{Hom}(X, \mathbb{Z}/m) \simeq \mathrm{disc}\, R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m)$$

hence

$$i\, R\mathrm{Hom}(X, \mathbb{Z}/m) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m).$$

We obtain

$$\begin{aligned}
X \widehat{\underline{\otimes}} \widehat{\mathbb{Z}} &:= (i(\mathrm{colim}\, R\mathrm{Hom}(X, \mathbb{Z}/m)))^D \\
&\simeq (\mathrm{colim}\, (i\, R\mathrm{Hom}(X, \mathbb{Z}/m)))^D \\
&\simeq \lim \left(i\, R\mathrm{Hom}(X, \mathbb{Z}/m))^D \right) \\
&\simeq \lim \left(R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m))^D \right) \\
&\simeq \lim \left(i(X) \underline{\otimes}^L \mathbb{Z}/m \right)
\end{aligned}$$

since the left adjoint functor i commutes with arbitrary colimits, and since $(-)^D$ transforms colimits into limits. Hence we have

$$\begin{aligned}
\mathrm{disc}(X \widehat{\underline{\otimes}} \widehat{\mathbb{Z}}) &\simeq \mathrm{disc}\left(\lim(i(X) \underline{\otimes}^L \mathbb{Z}/m) \right) \\
&\simeq \lim \left(\mathrm{disc}(i(X) \underline{\otimes}^L \mathbb{Z}/m) \right) \\
&\simeq \lim \left(\mathrm{Cofib}(\mathrm{disc} \circ i(X) \xrightarrow{m} \mathrm{disc} \circ i(X)) \right) \\
&\simeq \lim \left(X \otimes^L \mathbb{Z}/m \right)
\end{aligned}$$

since the right adjoint functor disc commutes with arbitrary limits. □

Remark 2.9. *Suppose that $X \in \mathbf{D}^b(\mathrm{Ab})$ is such that $X \otimes^L \mathbb{Z}/m$ is a perfect complex for any $\mathbb{Z}/m\mathbb{Z}$-modules. Then $R\underline{\mathrm{Hom}}(i(X), \mathbb{Z}/m)$ is discrete.*

Remark 2.10. *We have*

$$X \widehat{\underline{\otimes}} \widehat{\mathbb{Z}} \simeq R\underline{\mathrm{Hom}} \left(\mathrm{hocolim}\, R\mathrm{Hom}(X \otimes^L \mathbb{Z}/m, \mathbb{Q}/\mathbb{Z}), \mathbb{R}/\mathbb{Z} \right).$$

Lemma 2.11. *We have a canonical map $X \to X \widehat{\underline{\otimes}} \widehat{\mathbb{Z}}$ in $\mathbf{D}^b(\mathrm{LCA})$.*

Proof. The composite is an evident map

$$\begin{aligned}
i(R\mathrm{Hom}(X, \mathbb{Z}/m)) \xrightarrow{\sim} i \circ \mathrm{disc}(R\underline{\mathrm{Hom}}(X, \mathbb{Z}/m)) &\to R\underline{\mathrm{Hom}}(X, \mathbb{Z}/m)) \\
&\to R\underline{\mathrm{Hom}}(X, \mathbb{R}/\mathbb{Z}) \simeq C^D
\end{aligned}$$

inducing

$$i(\mathrm{colim}\, R\mathrm{Hom}(X, \mathbb{Z}/m)) \simeq \mathrm{colim}\, i(R\mathrm{Hom}(X, \mathbb{Z}/m)) \to X^D.$$

We obtain

$$X \xrightarrow{\sim} X^{DD} \to (i(\mathrm{colim}\, R\mathrm{Hom}(X, \mathbb{Z}/m)))^D =: X\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}.$$

□

Remark 2.12. *Let* $X \in \mathbf{D}^b(\mathrm{Ab})$ *such that its image in* $\mathbf{D}^b(\mathrm{LCA})$ *belongs to* $\mathbf{D}^b(\mathrm{FLCA})$*. Then one may consider* $X\underline{\otimes}^L\widehat{\mathbb{Z}}$ *and* $X\underline{\otimes}^L\widehat{\mathbb{Z}}_p$ *where* $\underline{\otimes}^L$ *is the tensor product in* $\mathbf{D}^b(\mathrm{FLCA})$*. There are canonical maps* $X\underline{\otimes}^L\widehat{\mathbb{Z}} \to X\widehat{\underline{\otimes}}^L\widehat{\mathbb{Z}}$ *and* $X\underline{\otimes}^L\mathbb{Z}_p \to X\widehat{\underline{\otimes}}^L\mathbb{Z}_p$ *but those maps are not equivalences in general. For example, we have*

$$\mathbb{Q}_p/\mathbb{Z}_p\underline{\otimes}^L\mathbb{Z}_p \simeq \mathbb{Q}_p/\mathbb{Z}_p$$

while

$$\mathbb{Q}_p/\mathbb{Z}_p\widehat{\underline{\otimes}}^L\mathbb{Z}_p \simeq \mathbb{Z}_p[1].$$

3. The complexes $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$ in $\mathbf{D}^b(\mathrm{LCA})$.

In this section we give a definition of $\mathbf{R\Gamma}_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$ assuming that the pair $(\mathcal{X}, n)$ satisfies Hypothesis 3.1 below. Hypothesis 3.1 is known for $n = 0, 1$ and arbitrary $\mathcal{X}$, hence our definition is unconditional in those cases.

Let p be a prime number, let $K/\mathbb{Q}_p$ be a finite extension, and let $\bar{K}/K$ be an algebraic closure. We denote by K^{un} the maximal unramified extension of K inside $\bar{K}$. Let X be a connected $(d-1)$-dimensional smooth proper scheme over K. Suppose that X has a proper regular model $\mathcal{X}/\mathcal{O}_K$, and let $\mathcal{X}_s$ be its special fiber, where $s \in \mathrm{Spec}(\mathcal{O}_K)$ is the closed point. We consider the following diagram.

$$\begin{array}{ccccc}
\mathcal{X}_{K^{un}} & \xrightarrow{\bar{j}} & \mathcal{X}_{\mathcal{O}_{K^{un}}} & \xleftarrow{\bar{i}} & \mathcal{X}_{\bar{s}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{X}_K & \xrightarrow{j} & \mathcal{X} & \xleftarrow{i} & \mathcal{X}_s
\end{array}$$

For any $n \geq 0$, we denote by $\mathbb{Z}(n)$ Bloch's cycle complex, which we consider as a complex of étale sheaves. We denote by $G_{\kappa(s)} \simeq \widehat{\mathbb{Z}}$ and by $W_{\kappa(s)} \simeq \mathbb{Z}$ the Galois group and the Weil group of the finite field $\kappa(s)$. We define

$$R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(\mathcal{X}_{K^{un}}, \mathbb{Z}(n))),$$

$$R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(n))),$$

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(n)).$$

There is a fiber sequence

$$R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(n)) \to R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(n)) \to R\Gamma_{et}(\mathcal{X}_{K^{un}}, \mathbb{Z}(n)). \tag{7}$$

Applying $R\Gamma(W_{\kappa(s)}, -)$ to (7) we obtain the fiber sequence

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_W(\mathcal{X}_K, \mathbb{Z}(n)). \tag{8}$$

Applying the natural transformation $R\Gamma(G_{\kappa(s)},-) \to R\Gamma(W_{\kappa(s)},-)$ to (7), we obtain the morphism of fiber sequences

$$\begin{array}{ccccc} R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n)) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) & \longrightarrow & R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_W(\mathcal{X}_K,\mathbb{Z}(n)) \end{array}$$

Recall from [8] the definition of eh-motivic cohomology, which we denote by $R\Gamma_{eh}(-,\mathbb{Z}(n))$.

Hypothesis 3.1. *We have a reduction map*

$$\bar{i}^* : R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}},\mathbb{Z}(n)) \to R\Gamma_{eh}(\mathcal{X}_{\bar{s}},\mathbb{Z}(n)).$$

Moreover, the complexes $R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))$, $R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n))$ and $R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}(n))$ are cohomologically bounded.

Following [8], we define Wh-motivic cohomology as follows:

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) := R\Gamma(W_{\kappa(s)}, R\Gamma_{eh}(\mathcal{X}_{\bar{s}},\mathbb{Z}(n))).$$

Definition 3.2. *Under hypothesis 3.1, we have a map*

$$R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n)) \tag{9}$$

and we denote by $C(\mathcal{X},n)$ its cofiber, so that we have a cofiber sequence

$$R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n)) \to C(\mathcal{X},n). \tag{10}$$

Similarly, we have a map

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) \tag{11}$$

and we denote by $C_W(\mathcal{X},n)$ its cofiber, so that we have a cofiber sequence

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) \to C_W(\mathcal{X},n). \tag{12}$$

Lemma 3.3. *Assume Hypothesis 3.1. Then there is a morphism $C(\mathcal{X},n) \to C_W(\mathcal{X},n)$ inducing equivalences*

$$C(\mathcal{X},n)\widehat{\otimes}\widehat{\mathbb{Z}} \xrightarrow{\sim} C_W(\mathcal{X},n)\widehat{\otimes}\widehat{\mathbb{Z}}$$

$$C(\mathcal{X},n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \xrightarrow{\sim} C_W(\mathcal{X},n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}$$

in $\mathbf{D}^b(\mathrm{Ab})$ and $\mathbf{D}^b(\mathrm{LCA})$ respectively.

Proof. The complex $C(\mathcal{X},n)$ is bounded by Hypothesis 3.1, and it follows that $C_W(\mathcal{X},n)$ is bounded as well. The morphism of functors $R\Gamma(G_{\kappa(s)},-) \to R\Gamma(W_{\kappa(s)},-)$ applied to the map of Hypothesis 3.1 gives a morphism from (9) to (11), hence a morphism of cofiber sequences

$$\begin{array}{ccccc} R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n)) & \longrightarrow & C(\mathcal{X},n) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_W(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) & \longrightarrow & C_W(\mathcal{X},n) \end{array}$$

The left and the middle vertical arrows are equivalences with finite coefficients. It follows that we have

$$C(\mathcal{X},n)\otimes^L \mathbb{Z}/m\mathbb{Z}\simeq C_W(\mathcal{X},n)\otimes^L \mathbb{Z}/m\mathbb{Z}$$

for any m. Hence $C(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}$ and $C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}$ are equivalent by Remark 2.10. □

Recall that we have fiber sequences

$$R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n))\to C(\mathcal{X},n) \tag{13}$$

and

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\to R\Gamma_{et}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{et}(\mathcal{X}_K,\mathbb{Z}(n)). \tag{14}$$

Proposition 3.4. *Assume Hypothesis 3.1. Then there exist* $R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(n))\in\mathbf{D}^b(\mathrm{LCA})$ *and* $R\widehat{\Gamma}_{et}(\mathcal{X}_K,\mathbb{Z}(n))\in\mathbf{D}^b(\mathrm{LCA})$ *endowed with fiber sequences*

$$R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n))\to C(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \tag{15}$$

and

$$R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\to R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(n))\to R\widehat{\Gamma}_{et}(\mathcal{X}_K,\mathbb{Z}(n)) \tag{16}$$

in $\mathbf{D}^b(\mathrm{LCA})$.

Proof. The proof is similar to the proof of Proposition 3.5 below. □

Proposition 3.5. *Assume Hypothesis 3.1. Then there exist* $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\in\mathbf{D}^b(\mathrm{LCA})$ *and* $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))\in\mathbf{D}^b(\mathrm{LCA})$ *endowed with fiber sequences*

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n))\to C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \tag{17}$$

and

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n)) \tag{18}$$

in $\mathbf{D}^b(\mathrm{LCA})$.

Proof. Composing the morphism in $\mathbf{D}^b(\mathrm{Ab})$

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n))\to C_W(\mathcal{X},n)$$

and the morphism in $\mathbf{D}^b(\mathrm{LCA})$

$$C_W(\mathcal{X},n)\to C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}$$

we obtain

$$R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n))\to C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}. \tag{19}$$

We define $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))$ as the fiber of (19) so that there is a fiber sequence

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n))\to C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \tag{20}$$

in $\mathbf{D}^b(\mathrm{LCA})$. Lemma 2.11 gives a map from (12) to (20) hence a map

$$R\Gamma_W(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)).$$

Then we define $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))\in\mathbf{D}^b(\mathrm{LCA})$ as the cofiber of the composite map

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n))\to R\Gamma_W(\mathcal{X},\mathbb{Z}(n))\to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)).$$

□

Remark 3.6. *For $n = 0, 1$, Hypothesis 3.1 holds, so that $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(1))$ are unconditionally defined.*

Proposition 3.7. *The maps*

$$R\widehat{\Gamma}_{et}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{eh}(\mathcal{X}_s, \mathbb{Z})$$

and

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z})$$

are equivalences.

Proof. By proper base change, the maps

$$R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m) \to R\Gamma_{eh}(\mathcal{X}_s, \mathbb{Z}/m)$$

are equivalences, hence

$$C(\mathcal{X}, 0) \otimes^L \mathbb{Z}/m \simeq C_W(\mathcal{X}, 0) \otimes^L \mathbb{Z}/m \simeq 0.$$

Hence we have

$$C(\mathcal{X}, 0)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \simeq C_W(\mathcal{X}, 0)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \simeq 0.$$

The result then follows from Proposition 3.4 and Proposition 3.5.

□

3.1.

Proposition 3.8. *Assume Hypothesis 3.1. There are canonical maps of fiber sequences*

$$(13) \to (15) \to (17) \text{ and } (14) \to (16) \to (18).$$

In particular we have canonical maps

$$R\Gamma_{et}(?, \mathbb{Z}(n)) \to R\widehat{\Gamma}_{et}(?, \mathbb{Z}(n)) \to R\Gamma_{ar}(?, \mathbb{Z}(n)) \tag{21}$$

for $? = \mathcal{X}, \mathcal{X}_K$.

Proof. The morphisms (15) → (17) and (16) → (18) are induced by the morphism of functors $R\Gamma(G_{\kappa(s)}, -) \to R\Gamma(W_{\kappa(s)}, -)$. The morphisms (13) → (15) and (14) → (16) are induced by the map $C(\mathcal{X}, n) \to C(\mathcal{X}, n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}$. □

Definition 3.9. *Assume Hypothesis 3.1. We define*

$$D(\mathcal{X}_K, n) := \mathrm{Cofib}(\delta(X, n))[-2]$$

where $\delta(\mathcal{X}_K, n)$ is defined as the compoition

$$\delta(\mathcal{X}_K, n) : R\Gamma(\mathcal{X}_s, Ri^!\mathbb{Q}(n)) \to R\Gamma(\mathcal{X}, \mathbb{Q}(n)) \to R\Gamma_{eh}(\mathcal{X}_s, \mathbb{Q}(n)).$$

Theorem 3.10. *Assume Hypothesis 3.1. We have fiber sequences in $\mathbf{D}^b(\mathrm{LCA})$*

$$R\Gamma_{eh}(\mathcal{X}_s, \mathbb{Q}(n))[-2] \to R\widehat{\Gamma}_{et}(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n)) \tag{22}$$

$$D(\mathcal{X}_K, n) \to R\widehat{\Gamma}_{et}(\mathcal{X}_K, \mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n)) \tag{23}$$

where $R\Gamma_{eh}(\mathcal{X}_s, \mathbb{Q}(n))[-2]$ and $D(\mathcal{X}_K, n)$ are seen as discrete objects of $\mathbf{D}^b(\mathrm{LCA})$.

Proof. Recall that the fiber of the map $R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n)) \to R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n))$ is $R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Q}(n))[-2]$, and that the map $C(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \to C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}$ is an equivalence. We denote by F the fiber of the map $R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))$. We obtain the diagram

$$\begin{array}{ccccc}
F & \xrightarrow{\sim} & R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Q}(n))[-2] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n)) & \longrightarrow & C(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}} \\
\downarrow & & \downarrow & & \downarrow\simeq \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)) & \longrightarrow & C_W(\mathcal{X},n)\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}
\end{array}$$

where the rows and colons are fiber sequences. It follows that the upper left horizontal map is an equivalence, hence the left colon gives the fiber sequence (22). We now consider the fiber of the morphism of fiber sequences (16) $\to$ (18). It yields the diagram

$$\begin{array}{ccccc}
R\Gamma(\mathcal{X}_s,Ri^!\mathbb{Q}(n)))[-2] & \longrightarrow & R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) & \longrightarrow & R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \\
\downarrow ? & & \downarrow & & \downarrow \\
R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Q}(n))[-2] & \longrightarrow & R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n)) \\
\downarrow & & \downarrow & & \downarrow \\
D'(\mathcal{X}_K,n) & \longrightarrow & R\widehat{\Gamma}_{et}(\mathcal{X}_K,\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))
\end{array}$$

where the rows and colons are fiber sequences. Here $D'(\mathcal{X}_K,n)$ is defined as the cofiber of the upper left vertical morphism ?. In order to obtain the fiber sequence (23) we need to identify the map ?. We consider the diagram with exact rows:

$$\begin{array}{ccccc}
R\Gamma(\mathcal{X}_s,Ri^!\mathbb{Q}(n)))[-2] & \longrightarrow & R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}(n))) & \longrightarrow & R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(n)) \\
\downarrow ? & & \downarrow & & \downarrow \\
R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Q}(n))[-2] & \longrightarrow & R\widehat{\Gamma}_{et}(\mathcal{X}_{\mathcal{O}_K},\mathbb{Z}(n)) & \longrightarrow & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))) \\
\downarrow\simeq & & \downarrow & & \downarrow \\
R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Q}(n))[-2] & \longrightarrow & R\Gamma_{eh}(\mathcal{X}_s,\mathbb{Z}(n))) & \longrightarrow & R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(n)))
\end{array}$$

where the composition of the left vertical maps is (equivalent to) $\delta(\mathcal{X}_K,n)$ and the lower left vertical map is an equivalence. Hence we have an equivalence $? \simeq \delta(\mathcal{X}_K,n)[-2]$ and therefore

$$D'(\mathcal{X}_K,n) \simeq D(\mathcal{X}_K,n)$$

We obtain (23).

$\square$

Corollary 3.11. *Suppose that the complexes $R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(n))$, $R\Gamma_{eh}(\mathcal{X}_s, \mathbb{Z}/m(n))$ and $R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m(n))$ are perfect complexes of $\mathbb{Z}/m\mathbb{Z}$-modules for any m. Then the maps (21) induce equivalences*

$$R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(n))\widehat{\otimes}\widehat{\mathbb{Z}} \xrightarrow{\sim} R\widehat{\Gamma}_{et}(\mathcal{X}, \mathbb{Z}(n))^{\delta}\widehat{\otimes}\widehat{\mathbb{Z}} \xrightarrow{\sim} R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(n))^{\delta}\widehat{\otimes}\widehat{\mathbb{Z}};$$

$$R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}(n))\widehat{\otimes}\widehat{\mathbb{Z}} \xrightarrow{\sim} R\widehat{\Gamma}_{et}(\mathcal{X}_K, \mathbb{Z}(n))^{\delta}\widehat{\otimes}\widehat{\mathbb{Z}} \xrightarrow{\sim} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))^{\delta}\widehat{\otimes}\widehat{\mathbb{Z}}$$

where $(-)^{\delta}$ denotes the functor $\mathrm{disc} : \mathbf{D}^b(\mathrm{LCA}) \to \mathbf{D}^b(\mathrm{Ab})$.

Proof. By Lemma 2.8 we have equivalences

$$\left(C(\mathcal{X}, n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}\right)^{\delta}\widehat{\otimes}\widehat{\mathbb{Z}} \to \left(C_W(\mathcal{X}, n)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}\right)^{\delta}\widehat{\otimes}\widehat{\mathbb{Z}} \simeq \left(C(\mathcal{X}, n)\widehat{\otimes}\widehat{\mathbb{Z}}\right)\widehat{\otimes}\widehat{\mathbb{Z}} \simeq C(\mathcal{X}, n)\widehat{\otimes}\widehat{\mathbb{Z}}.$$

The result then follows from Proposition 3.8 and Theorem 3.10. ☐

4. Duality for strict normal crossing schemes over finite fields

4.1. Wh-cohomology of strict normal crossing schemes.

Definition 4.1. *Let k be a field and let Y be a separated scheme of finite type over k. Then Y is said to be a* strict normal crossing scheme *if it is étale locally isomorphic to*

$$\mathrm{Spec}(k[T_0, \cdots, T_N]/(T_0 \cdots T_N))$$

such that any irreducible component of Y is smooth.

Fix a base field k and let $Y = \bigcup Y_i$ be a strict normal crossing scheme over k with components $Y_i, 1 \leq i \leq c$, and $Y^{(r)} = \coprod_{I\subseteq\{1,\dots,c\},|I|=r} Y_I$, where $Y_I = \bigcap_{i\in I} Y_i$. Consider the category $\mathcal{C}$ with objects smooth and proper schemes and morphisms $\mathrm{Hom}_{\mathcal{C}}(S, T)$ the free abelian group on scheme morphisms. Then we obtain a complex

$$Y^{(*)} = Y^{(0)} \leftarrow Y^{(1)} \leftarrow \cdots \leftarrow Y^{(d)},$$

in the category $\mathcal{C}$ where the maps $Y^{(r)} \to Y^{(r-1)}$ are given on components by

$$(-1)^k \cdot \mathrm{incl} : Y_{(i_1,\dots,i_r)} \to Y_{(i_1,\dots,\widehat{i_k},\dots,i_r)}. \tag{24}$$

Following [9], we denote by $\mathbb{Z}^c(n)$ the homological version of the cycle complex.

Proposition 4.2. *a) Let $R\Gamma_{eh}(Y^*, \mathbb{Z})$ be the total complex to the double complex $R\Gamma_{eh}(Y^{(*)}, \mathbb{Z})$, where the maps are induced by pull-back along* (24)*. Then the natural map*

$$R\Gamma_{eh}(Y, \mathbb{Z}) \to R\Gamma_{eh}(Y^*, \mathbb{Z})$$

is an equivalence.

b) Let $R\Gamma_{et}(Y^*, \mathbb{Z}^c(0))$ be the total complex to the double complex $R\Gamma_{et}(Y^{(*)}, \mathbb{Z}^c(0))$, where the maps are induced by push-forward along (24)*. Then the natural map*

$$R\Gamma_{et}(Y^*, \mathbb{Z}^c(0)) \to R\Gamma_{et}(Y, \mathbb{Z}^c(0))$$

is an equivalence.

Proof. The proof proceeds by induction on the number of irreducible components c. There is nothing to show if $c = 1$. Let $I' = I - \{c\}$ and $\hat{Y}$ be the normal crossing scheme built from the components of I'. Then we obtain an abstract blow up square where $\hat{Y}_c = Y_c \cap \hat{Y}$:

$$\begin{array}{ccc} \hat{Y}_c & \xrightarrow{\hat{i}_c} & \hat{Y} \\ \downarrow{\scriptstyle -\tilde{i}} & & \downarrow{\scriptstyle \hat{i}} \\ Y_c & \xrightarrow{i_c} & Y \end{array}$$

In case a), by eh-descent the natural map from $R\Gamma_{eh}(Y, \mathbb{Z})$ to the shifted cone of

$$R\Gamma_{eh}(Y_c, \mathbb{Z}) \oplus R\Gamma_{eh}(\hat{Y}, \mathbb{Z}) \stackrel{(-\tilde{i},\hat{i}_c)}{\longrightarrow} R\Gamma_{eh}(\hat{Y}_c, \mathbb{Z})$$

is an equivalence. By induction this is quasi-isomorphic to the shifted cone of double-complexes

$$R\Gamma_{eh}(Y_c, \mathbb{Z}) \oplus R\Gamma_{eh}(\hat{Y}^*, \mathbb{Z}) \stackrel{(-\tilde{i},\hat{i}_c)}{\longrightarrow} R\Gamma_{eh}(\hat{Y}_c^*, \mathbb{Z})$$

and we claim that this is the same complex as $R\Gamma_{eh}(Y^*, \mathbb{Z})$. The complexes are the same because $Y^{(0)} = Y_c \coprod \hat{Y}^{(0)}$, and $Y^{(i)} = \hat{Y}^{(i)} \coprod \hat{Y}_c^{(i-1)}$ is the disjoint decomposition into those intersections which are not contained in Y_c and those which are. To see that the maps are the same, note that the map $-\tilde{i}$ from Y_c is the same as the map in $-\mathrm{incl} : Y_{i_1,c} \to Y_c$ appearing in Y^*, and the other maps are also easily seen to be the same.

The proof of b) is the same after reversing all arrows of cohomology groups and replacing eh-descent by the localization property, which holds for etale hypercohomology by [9]. □

Definition 4.3. *For a strict normal crossing scheme Y over a finite field k we define $R\Gamma_W(Y^*, \mathbb{Z})$ as the hypercohomology of $R\Gamma(W_k, R\Gamma_{et}(\bar{Y}^*, \mathbb{Z}))$, where $\bar{Y}^*$ is the base change of Y^* to the algebraic closure. Similarly, we define $R\Gamma_{Wh}(Y^*, \mathbb{Z})$ as the hypercohomology of $R\Gamma(W_k, R\Gamma_{eh}(\bar{Y}^*, \mathbb{Z}))$.*

Of course, $R\Gamma_W(Y^*, \mathbb{Z})$ (respectively $R\Gamma_{Wh}(Y^*, \mathbb{Z})$) may also be defined as the total complex of the double complex $R\Gamma_W(Y^{(*)}, \mathbb{Z})$ (respectively $R\Gamma_{Wh}(Y^{(*)}, \mathbb{Z})$).

Corollary 4.4. *Let Y be a strict normal crossing scheme over a finite field k. There is a canonical map*

$$R\Gamma_W(Y^*, \mathbb{Z}) \to R\Gamma_{Wh}(Y, \mathbb{Z}).$$

Moreover, this map is an equivalence if resolution of singularities for schemes over k of dimension $\leq \dim(Y)$ holds.

Proof. We consider the commutative square

$$\begin{array}{ccc} R\Gamma_W(Y, \mathbb{Z}) & \longrightarrow & R\Gamma_W(Y^*, \mathbb{Z}) \\ \downarrow & & \downarrow \\ R\Gamma_{Wh}(Y, \mathbb{Z}) & \xrightarrow{\sim} & R\Gamma_{Wh}(Y^*, \mathbb{Z}) \end{array}$$

where the lower horizontal map is an equivalence by Proposition 4.2. This defines the map $R\Gamma_W(Y^*,\mathbb{Z}) \to R\Gamma_{Wh}(Y,\mathbb{Z})$. Moreover, assuming resolution of singularities for schemes over k of dimension $\leq \dim(Y)$, the right vertical map is an equivalence by [8, Thm. 4.3], since the schemes $Y^{(i)}$ are smooth. The result follows.

□

Notation 4.5. *If Y is a scheme over a finite field such that the reduced scheme Y^{red} is a strict normal scheme. Then we set*

$$R\Gamma_W(Y^*,\mathbb{Z}) := R\Gamma_W((Y^{\mathrm{red}})^*,\mathbb{Z}).$$

4.2. Duality.

Theorem 4.6. *Let Y be a strict normal crossing scheme of over a finite field k such that $R\Gamma_W(Y,\mathbb{Z}^c(0))$ is a perfect complex of abelian groups. Then there is a perfect pairing*

$$R\Gamma_W(Y^*,\mathbb{Z}) \otimes^L R\Gamma_W(Y,\mathbb{Z}^c(0)) \to \mathbb{Z}[-1] \tag{25}$$

of perfect complexes of abelian groups.

Proof. We first consider the case where Y is smooth and proper. Let $f : Y \to \mathrm{Spec}(k) = s$ be the structure morphism. The push-forward map

$$Rf_*\mathbb{Z}^c(0)^Y \to \mathbb{Z}^c(0)^s \simeq \mathbb{Z}[0]$$

induces a trace map

$$R\Gamma_W(Y,\mathbb{Z}^c(0)) \to R\Gamma_W(s,\mathbb{Z}) \to \mathbb{Z}[-1].$$

We consider the map

$$R\Gamma_W(Y,\mathbb{Z}) \longrightarrow R\mathrm{Hom}(R\Gamma_W(Y,\mathbb{Z}^c(0)),\mathbb{Z}[-1]) \tag{26}$$

induced by the pairing

$$R\Gamma_W(Y,\mathbb{Z}) \otimes^L R\Gamma_W(Y,\mathbb{Z}^c(0)) \to R\Gamma_W(Y,\mathbb{Z}^c(0)) \to R\Gamma_W(s,\mathbb{Z}^c(0)) \to \mathbb{Z}[-1].$$

In order to show that the morphism of perfect complexes (26) is an equivalence, it is enough to show that (26) $\otimes^L \mathbb{Z}/m\mathbb{Z}$ is an equivalence for any integer m. But (26) $\otimes^L \mathbb{Z}/m\mathbb{Z}$ may be identified with the map

$$R\Gamma_{et}(Y,\mathbb{Z}/m\mathbb{Z}) \longrightarrow R\mathrm{Hom}(R\Gamma_{et}(Y,\mathbb{Z}^c(0)/m),\mathbb{Q}/\mathbb{Z}[-1])$$

which is an equivalence by [9, Theorem 5.1]. Hence (26) is an equivalence.

Suppose now that Y is a strict normal crossing scheme. The equivalence (26) is functorial with respect to closed immersions, hence we obtain a morphism of double complexes

$$R\Gamma_W(Y^{(*)},\mathbb{Z}) \longrightarrow R\mathrm{Hom}(R\Gamma_W(Y^{(*)},\mathbb{Z}^c(0)),\mathbb{Z}[-1])$$

which induces an equivalence of total complexes

$$R\Gamma_W(Y^*,\mathbb{Z}) \longrightarrow R\mathrm{Hom}(R\Gamma_W(Y^*,\mathbb{Z}^c(0)),\mathbb{Z}[-1]) \xleftarrow{\sim} R\mathrm{Hom}(R\Gamma_W(Y,\mathbb{Z}^c(0)),\mathbb{Z}[-1]),$$

where the last equivalence follows from Proposition 4.2(b).

□

5. Working definitions for the Tate twists $n = 0, d$

The aim of this section is to give an alternative definition of $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(0))$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$ for $\mathcal{X}$ of pure dimension d, which is expected to coincide with the conditional definition given Section 3.

In order to redefine $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(0))$, we replace $R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z})$ by $R\Gamma_W(\mathcal{X}_s^*, \mathbb{Z})$ and we apply the construction of Section 3. In view of Corollary 4.4, the two construction agree provided that resolution of singularities holds for schemes of dimension $\leq \dim(\mathcal{X}_s)$.

In order to redefine $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$, we again apply the construction of Section 3 except that we complete the entire complex $R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$ instead of $C_W(\mathcal{X}, n)$, so that the existence of the reduction map $R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}(d))$ of Hypothesis 3.1 is no longer required. The two construction agree provided that the cohomology of the complex $R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}(d))$ consists of cohomology groups.

5.1. Working definition for $n = 0$. If $\mathcal{X}_s$ is a strict normal crossing scheme, we replace $R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z})$ with $R\Gamma_W(\mathcal{X}_s^*, \mathbb{Z})$ where we use the notation of Definition 4.3 and Notation 4.5. There is a canonical map

$$R\Gamma_W(\mathcal{X}, \mathbb{Z}) \to R\Gamma_W(\mathcal{X}_s, \mathbb{Z}) \to R\Gamma_W(\mathcal{X}_s^*, \mathbb{Z})$$

whose cofiber we still denote by $C_W(\mathcal{X}, 0)$. Then we define $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \in \mathbf{D}^b(\mathrm{LCA})$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \in \mathbf{D}^b(\mathrm{LCA})$ endowed with fiber sequences

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_W(\mathcal{X}_s^*, \mathbb{Z}) \to C_W(\mathcal{X}, 0)\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \tag{27}$$

and

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \tag{28}$$

in $\mathbf{D}^b(\mathrm{LCA})$.

Remark 5.1. *Suppose that resolution of singularities for schemes over $\kappa(s)$ of dimension $\leq d-1$ holds. It follows from Corollary 4.4 that if $\mathcal{X}_s$ is a strict normal crossing scheme, then the complexes $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})$ defined in (27) and (28) are equivalent to the complexes defined in Section 3.*

5.2. Working definition for $n = d$. Hypothesis 3.1 is not known for $n > 1$ in general. However, the cohomology $R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}(d))$ is expected to consist of finite abelian groups. Therefore, under the assumption that $R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ is cohomologically bounded, we may redefine arithmetic cohomology with coefficients in $\mathbb{Z}(d)$ as follows.

The complex $R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$ is not known to be bounded below. However the complex

$$R\Gamma_W(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d)) \simeq R\Gamma_{et}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(d))$$

is bounded, hence the cohomology groups $H^i_W(\mathcal{X}, \mathbb{Z}(d))$ are $\mathbb{Q}$-vector spaces for $i << 0$. In particular, for $a < b << 0$ the map

$$\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to \tau^{>b} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d))$$

induces an equivalence

$$(\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}} \xrightarrow{\sim} (\tau^{>b} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)))\underline{\widehat{\otimes}}\widehat{\mathbb{Z}}.$$

Definition 5.2. *Let $a << 0$. We define*

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) := (\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)))\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}.$$

If $R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ is cohomologically bounded, we define $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ as the cofiber of the composite map

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \to \tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$$

in $\mathbf{D}^b(\mathrm{LCA})$. Similarly, we define

$$R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(d)) := (\tau^{>a}R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)))\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}.$$

If $R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ is cohomologically bounded, we define $R\widehat{\Gamma}_{et}(\mathcal{X}_K,\mathbb{Z}(d))$ as the cofiber of the composite map

$$R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \to \tau^{>a}R\Gamma_{et}(\mathcal{X},\mathbb{Z}(d)) \to R\widehat{\Gamma}_{et}(\mathcal{X},\mathbb{Z}(d))$$

in $\mathbf{D}^b(\mathrm{LCA})$.

The proof of Theorem 3.10 gives a cofiber sequence

$$R\Gamma(\mathcal{X}_s, Ri^!\mathbb{Q}(d))[-1] \to R\widehat{\Gamma}_{et}(\mathcal{X}_K,\mathbb{Z}(d)) \to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)). \tag{29}$$

Remark 5.3. *Suppose that $\mathcal{X}$ satisfies Hypothesis 3.1 and suppose that the complex $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z}(d))$ is bounded with finite cohomology groups. Then the complexes of Definition 5.2 are equivalent to the complexes of Definition 3.5.*

5.3. Finite ranks.

Proposition 5.4. *Assume resolution of singularities for schemes over $\kappa(s)$ of dimension $\leq d-1$. Then $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z})$ is a perfect complex of abelian groups.*

Proof. By [8, Lemma 2.7], and using Wh-cohomology with compact support, one is reduced to show that $R\Gamma_{Wh}(Y,\mathbb{Z})$ is perfect for any smooth projective scheme Y over $\kappa(s)$ of dimension $\leq d-1$. By [8, Corollary 5.5], the map $R\Gamma_W(Y,\mathbb{Z}) \to R\Gamma_{Wh}(Y,\mathbb{Z})$ is an equivalence for smooth projective Y. The fact that $R\Gamma_W(Y,\mathbb{Z})$ is perfect was observed in [16].

□

Proposition 5.5. *Suppose that $\mathcal{X}_s$ is a normal crossing scheme. Then $R\Gamma_W(\mathcal{X}_s^*,\mathbb{Z})$ is a perfect complex of abelian groups.*

Proof. We set $Y = \mathcal{X}_s$. For any $0 \leq i \leq d$, the scheme $Y^{(i)}$ is smooth proper, hence $R\Gamma_W(Y^{(i)},\mathbb{Z})$ is perfect. It follows that the total complex $R\Gamma_W(Y^*,\mathbb{Z})$ of the double complex $R\Gamma_W(Y^{(*)},\mathbb{Z})$ is perfect as well. □

Proposition 5.6.
(1) *Assume that $\mathcal{X}_s$ is a strict normal crossing scheme or assume resolution of singularities for schemes over $\kappa(s)$ of dimension $\leq d-1$. Then $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})$ belong to $\mathbf{D}^b(\mathrm{FLCA})$.*
(2) *Assume that $R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ is a perfect complex of abelian groups. Then $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$ and $R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))$ belong to $\mathbf{D}^b(\mathrm{FLCA})$.*

Proof. Under these hypothesis, the complexes $R\Gamma_W(\mathcal{X}_s^*,\mathbb{Z})$ (resp. $R\Gamma_{Wh}(\mathcal{X}_s,\mathbb{Z})$) and $R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ are perfect complexes of abelian groups by Proposition 5.5, Proposition 5.4 and by Hypothesis 5.8, hence they belong to $\mathbf{D}^b(\mathrm{FLCA})$

by Lemma 2.2. By the proof of Proposition 6.3, $R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d))$ is (up to a shift) dual to $R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Q}/\mathbb{Z})$. Hence, by Lemma 2.2, it is enough to check that $H^j_{et}(\mathcal{X}_s, Ri^!\mathbb{Q}/\mathbb{Z})$ is an abelian group of finite ranks for all $j \in \mathbb{Z}$. Since $H^j_{et}(\mathcal{X}_s, Ri^!\mathbb{Q}/\mathbb{Z})$ is torsion and discrete, it is both of finite $\mathbb{Z}$-rank and of finite $\mathbb{S}^1$-rank. It remains to see that it is of finite p-rank for any prime number p. This follows from the fact that $H^j_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/p\mathbb{Z})$ is a finite group for any $j \in \mathbb{Z}$ by [9, Theorem 7.5]. □

5.4. Locally compact cohomology groups. The following condition is borrowed from [24].

Hypothesis 5.7. *The scheme $(\mathcal{X}_s)^{\mathrm{red}}$ is a strict normal scheme over $\kappa(s)$ and $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K) = S$ is log-smooth with respect to the log-structures associated with $(\mathcal{X}_s)^{\mathrm{red}}$ and s respectively.*

The following hypothesis holds if $d \leq 2$. It is conjecturally true in general.

Hypothesis 5.8. *The complex $R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ is a perfect complex of abelian groups.*

Theorem 5.9. (1) *Suppose that $(\mathcal{X}_s)^{\mathrm{red}}$ is a strict normal crossing scheme. Then for any $i \in \mathbb{Z}$, the object $H^i_{ar}(\mathcal{X}_K, \mathbb{Z}) \in \mathcal{LH}(\mathrm{LCA})$ is a discrete abelian group. More precisely, $H^j_{ar}(\mathcal{X}_K, \mathbb{Z}) \in \mathcal{LH}(\mathrm{LCA})$ is an extension of a torsion abelian group by a finitely generated abelian group.*
(2) *Suppose that $\mathcal{X}$ satisfies both Hypothesis 5.7 and Hypothesis 5.8. Then for any $i \in \mathbb{Z}$, the object $H^i_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$ is a locally compact abelian group. More precisely, $H^i_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$ is an extension of a finitely generated abelian group by a finitely generated $\mathbb{Z}_p$-module endowed with the p-adic topology.*

Proof. We have a long exact sequence in the abelian category $\mathcal{LH}(\mathrm{LCA})$

$$H^j_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \to H^j_{ar}(\mathcal{X}, \mathbb{Z}) \to H^j_{ar}(\mathcal{X}_K, \mathbb{Z}) \to H^{j+1}_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \to H^{j+1}_{ar}(\mathcal{X}, \mathbb{Z})$$

where $H^j_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \simeq H^{j-1}_W(\mathcal{X}_s, Ri^!\mathbb{Q}/\mathbb{Z})$ is a discrete torsion abelian group (see the proof of Proposition 6.3) and $H^j_{ar}(\mathcal{X}, \mathbb{Z}) \simeq H^j_W(\mathcal{X}^*_s, \mathbb{Z})$ is a discrete finitely generated abelian group by Proposition 5.5. Hence $H^j_{ar}(\mathcal{X}_K, \mathbb{Z}) \in \mathcal{LH}(\mathrm{LCA})$ is an extension of a torsion abelian group by a finitely generated abelian group, since the subcategory $\mathrm{LCA} \subset \mathcal{LH}(\mathrm{LCA})$ is stable by extensions.

We prove (2). We have a long exact sequence in $\mathcal{LH}(\mathrm{LCA})$

$$H^j_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \to H^j_{ar}(\mathcal{X}, \mathbb{Z}(d)) \to H^j_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \to H^{j+1}_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$$

where $H^j_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))$ is a discrete finitely generated abelian group by Hypothesis 5.8. Moreover, $H^j_{ar}(\mathcal{X}, \mathbb{Z}(d)) \in \mathcal{LH}(\mathrm{LCA})$ is the group

$$H^j_{et}(\mathcal{X}, \widehat{\mathbb{Z}}(d)) \simeq \prod_l H^j_{et}(\mathcal{X}, \mathbb{Z}_l(d)) := \prod_l H^j(R\lim(R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/l^\bullet))$$

where the product is taken over the set of prime numbers l, and the finitely generated $\mathbb{Z}_l$-module $H^j_{et}(\mathcal{X}, \mathbb{Z}_l(d))$ is endowed with the l-adic topology. We need to show that the image of the map $H^j_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \to H^j_{et}(\mathcal{X}, \widehat{\mathbb{Z}}(d))$ is finite. It will follow that $H^j_{ar}(\mathcal{X}_K, \mathbb{Z}(d))$ is an extension of a finitely generated abelian group by a profinite abelian group.

Since the group

$$H^j_{et}(\mathcal{X}, \widehat{\mathbb{Z}}'(d)) \simeq \prod_{l \neq p} H^j_{et}(\mathcal{X}, \mathbb{Z}_l(d))$$

is finite for any $j \in \mathbb{Z}$ by Lemma 5.11, and since we have an isomorphism of finitely generated $\mathbb{Z}_p$-modules

$$H^j_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq H^j_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}_p(d)),$$

it is enough to show that the image of the map

$$H^j_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}_p(d)) \to H^j_{et}(\mathcal{X}, \mathbb{Z}_p(d))$$

is finite. One is therefore reduced to show that the map

$$H^j_{et}(\mathcal{X}_s, Ri^!\mathbb{Q}_p(d)) \to H^j_{et}(\mathcal{X}, \mathbb{Q}_p(d))$$

is the zero-map, which follows from Theorem 5.10 by the localization sequence. □

The following result easily follows from Sato's work [24].

Theorem 5.10. *Suppose that $\mathcal{X}$ satisfies Hypothesis 5.7. Then for any $i \in \mathbb{Z}$, the map*

$$H^i_{et}(\mathcal{X}, \mathbb{Q}_p(d)) \to H^i_{et}(\mathcal{X}_K, \mathbb{Q}_p(d)) \tag{30}$$

is injective.

Proof. We first observe that we have isomorphisms

$$H^i_{et}(\mathcal{X}, \mathbb{Q}_p(d)) \simeq H^i_{et}(\mathcal{X}, \mathbb{Q}^S_p(d)) \tag{31}$$

compatible with the map (30), where

$$R\Gamma_{et}(\mathcal{X}, \mathbb{Q}^S_p(d)) := R\lim R\Gamma_{et}(\mathcal{X}, \mathfrak{T}_r(d)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is the complex studied in [24]. Indeed, this follows from the equivalences

$$R\Gamma(\mathcal{X}_{et}, \mathfrak{T}_r(d)) \simeq R\mathrm{Hom}(R\Gamma_{\mathcal{X}_s}(\mathcal{X}_{et}, \mathbb{Z}/p^r), \mathbb{Z}/p^r)[-2d-1] \tag{32}$$

$$\simeq R\Gamma(\mathcal{X}_{et}, \mathbb{Z}(d)/p^r), \tag{33}$$

given by [23, Thm 10.1.1] and [9, Proof of Thm 7.5], and from the fact that (30) is induced by the dual of the map

$$R\Gamma(\mathcal{X}_{K,et}, \mathbb{Z}/p^r)[-1] \to R\Gamma_{\mathcal{X}_s}(\mathcal{X}_{et}, \mathbb{Z}/p^r).$$

Hence we are reduced to show that the map

$$H^i_{et}(\mathcal{X}, \mathbb{Q}^S_p(d)) \to H^i_{et}(\mathcal{X}_K, \mathbb{Q}_p(d))$$

is injective. By [24, Prop. 3.4(1)], [24, Section 4.1] and [24, Thm. 5.3], there is a morphism of spectral sequences from

$$H^i_f(G_K, H^j_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))) \Rightarrow H^{i+j}_{et}(\mathcal{X}, \mathbb{Q}^S_p(d))$$

to

$$H^i(G_K, H^j_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))) \Rightarrow H^{i+j}_{et}(\mathcal{X}_K, \mathbb{Q}_p(d))$$

where the first spectral sequence degenerates into isomorphisms

$$H^j_{et}(\mathcal{X}, \mathbb{Q}_p^S(d)) \xrightarrow{\sim} H^1_f(G_K, H^{j-1}_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))).$$

Since we have [24, Prop. 5.10(1)]

$$H^0_f(G_K, H^j_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))) = H^0(G_K, H^j_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))) = 0$$

for any $j \in \mathbb{Z}$, we obtain a commutative square

$$\begin{array}{ccc} H^j_{et}(\mathcal{X}, \mathbb{Q}_p^S(d)) & \longrightarrow & H^j_{et}(\mathcal{X}_K, \mathbb{Q}_p(d)) \\ \downarrow \simeq & & \downarrow \\ H^1_f(G_K, H^{j-1}_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))) & \longrightarrow & H^1(G_K, H^{j-1}_{et}(\mathcal{X}_{\bar{K}}, \mathbb{Q}(d))) \end{array}$$

where the vertical maps are edge morphisms of the corresponding spectral sequences. Here the left vertical map is an isomorphism and the lower horizontal map is injective. It follows that the upper horizontal map is injective as well. □

Lemma 5.11. *Suppose that $\mathcal{X}$ satisfies Hypothesis 5.7. Then the group*

$$H^i_{et}(\mathcal{X}, \widehat{\mathbb{Z}}'(d)) := H^i(R\mathrm{lim}_{p\nmid m}\, R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \otimes^L \mathbb{Z}/m)$$

is finite for any $i \in \mathbb{Z}$.

Proof. We set $\mathbb{Z}/m(d) := \mathbb{Z}(d) \otimes^L \mathbb{Z}/m$. By duality and proper base change, we have

$$\begin{aligned} &(34) \quad & R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d)) &\simeq R\mathrm{Hom}(R\Gamma_{\mathcal{X}_s}(\mathcal{X}_{et}, \mathbb{Z}/m), \mathbb{Z}/p^r)[-2d-1] \\ &(35) & &\simeq R\Gamma_{et}(\mathcal{X}, \mu_m^{\otimes d}) \\ &(36) & &\simeq R\Gamma_{et}(\mathcal{X}_s, \mu_m^{\otimes d}) \end{aligned}$$

for any m prime to p. Moreover, we have

$$R\Gamma_{et}(\mathcal{X}_s, \mu_m^{\otimes d}) \simeq R\Gamma_{eh}(\mathcal{X}_s, \mu_m^{\otimes d}) \simeq R\Gamma_{eh}(\mathcal{X}_s^*, \mu_m^{\otimes d}) \simeq R\Gamma_{et}(\mathcal{X}_s^*, \mu_m^{\otimes d})$$

by [8, Theorem 3.6] and the argument of Proposition 4.2(a). Hence we have

$$R\Gamma_{et}(\mathcal{X}, \widehat{\mathbb{Z}}'(d)) \simeq R\mathrm{lim}_{p\nmid m}\, R\Gamma_{et}(\mathcal{X}_s^*, \mu_m^{\otimes d})$$

It remains to check that the cohomology of the right hand side is finite. For any i, we have $R\Gamma_{et}(\mathcal{X}_s^{(i)}, \mathbb{Q}_l(d)) \simeq 0$ by a weight argument. Hence the result follows from the fact [5] that the cohomology of $R\Gamma_{et}(\mathcal{X}_s^{(i)}, \mathbb{Z}_l(d))$ is finite for all $l \neq p$ and zero for almost all l. Note that $\mathcal{X}_s^{(i)}$ is smooth. □

6. Duality theorems

From now on, we use the notations and definitions introduced in Section 5.

6.1. Duality with $\mathbb{Z}$-coefficients.

Theorem 6.1. *Assume either $d \leq 2$ or that $(\mathcal{X}_s)^{\mathrm{red}}$ is a strict normal crossing scheme satisfying Hypothesis 5.8. Then there is a perfect pairing*

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \underline{\otimes}^L R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \longrightarrow \mathbb{Z}[-2d]$$

in $\mathbf{D}^b(\mathrm{FLCA})$.

The hypotheses of the theorem are assumed throughout its proof.

Proof. Recall from Proposition 5.6 that $R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(n))$ belongs to $\mathbf{D}^b(\mathrm{FLCA})$ for $n = 0, d$, so that the tensor product

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \underline{\otimes}^L R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})$$

defined in Section 2, makes sense. Moreover, the equivalence [9]

$$Ri^! \mathbb{Z}^c(0)^{\mathcal{X}} \simeq \mathbb{Z}^c(0)^{\mathcal{X}_s}$$

and the push-forward map

$$Rf_* \mathbb{Z}^c(0)^{\mathcal{X}_s} \to \mathbb{Z}^c(0)^s \simeq \mathbb{Z}[0]$$

give trace maps

$$R\Gamma_W(\mathcal{X}_s, Ri^! \mathbb{Z}(d)) \simeq R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c[-2d]) \to R\Gamma_W(s, \mathbb{Z}[-2d]) \to \mathbb{Z}[-2d-1]$$

and

$$R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}/m(d)) \to R\Gamma_{et}(s, \mathbb{Z}/m[-2d]) \to \mathbb{Z}/m[-2d-1].$$

We start with the following

Proposition 6.2. *The obvious product maps $\mathbb{Z} \otimes^L \mathbb{Z}(d) \to \mathbb{Z}(d)$, in the derived ∞-category of étale sheaves over $\mathcal{X}_{\mathcal{O}_{K^{un}}}$ and $\mathcal{X}_{K^{un}}$ induce perfect pairings*

$$R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}/m) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d)) \to R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1]$$

and

$$R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}/m(d)) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1]$$

for any m.

Proof. Consider the obvious product map $\mathbb{Z} \otimes^L \mathbb{Z}(d-n) \to \mathbb{Z}(d)$ in the derived ∞-category of étale sheaves over $\mathcal{X}$ and $\mathcal{X}$. We have a commutative square

$$\begin{array}{ccc} R\Gamma_{et}(\mathcal{X}, \mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \\ \downarrow & & \downarrow \\ R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}(d)) \end{array}$$

Taking the fibers of the vertical arrows gives the product map

$$R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}(d)) \to R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}(d)). \tag{37}$$

The product map

$$R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}(d)) \otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}) \to R\Gamma_{et}(\mathcal{X}_s, Ri^! \mathbb{Z}(d)) \tag{38}$$

is obtained similarly.

By [9, Theorem 7.5], the pairing

$$R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m)\otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m(d)) \to R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1],$$

induced by (37), is perfect. The pairing induced by (38)

$$R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m(d))\otimes^L R\Gamma_{et}(\mathcal{X}, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m(d)) \to \mathbb{Z}/m[-2d-1]$$

is perfect as well, since it reduces, by purity and proper base change, to

$$R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}^c/m[-2d])\otimes^L R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}/m) \to R\Gamma_{et}(\mathcal{X}_s, \mathbb{Z}^c/m[-2d]) \to \mathbb{Z}/m[-2d-1]$$

which is perfect by [9]. □

For $n = 0$ or $n = d$, consider the product map

$$R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(n)) \otimes^L R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(d-n)) \to R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d)).$$

This product map is induced by the obvious product maps $\mathbb{Z}\otimes^L\mathbb{Z}(d) \to \mathbb{Z}(d)$ in the derived ∞-category of étale sheaves over $\mathcal{X}_{\mathcal{O}_{K^{un}}}$ and $\mathcal{X}_{K^{un}}$, as in the proof of Proposition 6.2. Applying $R\Gamma(W_{\kappa(s)}, -)$ and composing with the trace map, we obtain

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(n)) \otimes^L R\Gamma_W(\mathcal{X}, \mathbb{Z}(d-n)) \to R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \to \mathbb{Z}[-2d-1].$$

This yields the morphisms

$$R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1])$$

which in turn induces

$$\tau^{>a}R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \tag{39}$$

for $a << 0$, since the right hand side is bounded. Similarly, we obtain

$$R\Gamma_W(\mathcal{X}, Ri^!\mathbb{Z}(d)) \to R\mathrm{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d-1]). \tag{40}$$

Composing (39) with the canonical map (see Lemma 2.6)

$$R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \to R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1])$$

we obtain

$$\tau^{>a}R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) \to R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]). \tag{41}$$

Proposition 6.3. *The map (41) factors through an equivalence*

$$R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}(d)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \tag{42}$$

in $\mathbf{D}^b(\mathrm{FLCA})$.

Proof. One has

$$\begin{aligned}
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) &:= \tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d))\widehat{\underline{\otimes}}\widehat{\mathbb{Z}}\\
&\simeq (\mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}))^D\\
&\simeq (\mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}))^D\\
&\simeq R\underline{\mathrm{Hom}}(\mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}[-2d-1]),\mathbb{R}/\mathbb{Z}[-2d-1])\\
&\stackrel{\sim}{\to} R\underline{\mathrm{Hom}}(\mathrm{hocolim}\, R\Gamma_{et}(\mathcal{X}_s,Ri^!\mathbb{Z}/m),\mathbb{R}/\mathbb{Z}[-2d-1])\\
&\simeq R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z}),\mathbb{R}/\mathbb{Z}[-2d-1])\\
&\simeq R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]),\mathbb{R}/\mathbb{Z}[-2d-1])\\
&\simeq R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1])
\end{aligned}$$

where we use Proposition 6.2, the vanishing

$$R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]),\mathbb{R})\simeq 0. \tag{43}$$

proven in Lemma 6.4 below and $Ri^!\mathbb{Q}\simeq 0$. This last observation follows from the fact $\mathbb{Q}\to Rj_*j^*\mathbb{Q}$ is an equivalence because $\mathcal{X}$ is normal. Hence the map (41) factors through an equivalence

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))\stackrel{\sim}{\to} R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]).$$

□

Lemma 6.4. *We have*

$$R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]),\mathbb{R})\simeq R\underline{\mathrm{Hom}}(\mathbb{R},R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1]))\simeq 0.$$

Proof. We first prove the first assertion. As observed above, we have $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1])\simeq R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$. Since $R\underline{\mathrm{Hom}}(\mathbb{R},-)$ and $R\underline{\mathrm{Hom}}(-,\mathbb{R})$ are exact functors, and using the t-structure on $\mathbf{D}^b(\mathrm{FLCA})$, we may suppose that $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Q}/\mathbb{Z})$ is cohomologically concentrated in one degree. Hence one is reduced to show that

$$R\underline{\mathrm{Hom}}(A,\mathbb{R})\simeq R\underline{\mathrm{Hom}}(\mathbb{R},A)\simeq 0$$

for any torsion discrete abelian group of finite ranks A. This follows from [13] Proposition 4.15. □

Corollary 6.5. *We have*

$$R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)),\mathbb{R})\simeq R\underline{\mathrm{Hom}}(\mathbb{R},R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)))\simeq 0.$$

Proof. In the proof of Proposition 6.3, we have shown that $R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))$ is, up to a shift, Pontryagin dual to $R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}[1])$. Hence the corollary follows from Lemma 6.4 since we have $R\underline{\mathrm{Hom}}(X,Y)\simeq R\underline{\mathrm{Hom}}(Y^D,X^D)$ for any $X,Y\in\mathbf{D}^b(\mathrm{FLCA})$. □

Similarly, we have the

Proposition 6.6. *The map*

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\to R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \tag{44}$$

factors through an equivalence

$$R\Gamma_W(\mathcal{X}_s,Ri^!\mathbb{Z}(d))\stackrel{\sim}{\longrightarrow} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]). \tag{45}$$

Proof. Over $\mathcal{X}$, one has $\mathbb{Z}^c(0) = \mathbb{Z}(d)[2d]$. Moreover, one has
$$Ri^!\mathbb{Z}^c(0) \simeq \mathbb{Z}^c(0)$$
hence
$$Ri^!\mathbb{Z}(d) = Ri^!\mathbb{Z}^c(0)[-2d] \simeq \mathbb{Z}^c(0)[-2d].$$
If $\mathcal{X}_s$ is a strict normal crossing scheme, we may therefore identify the map
$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d))[2d] \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d-1])[2d] \tag{46}$$
with the composite morphism
$$R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s^*, \mathbb{Z}), \mathbb{Z}[-1]) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s^*, \mathbb{Z}), \mathbb{Z}[-1])$$
which is an equivalence of perfect complexes of abelian groups by Hypothesis 5.8, Proposition 5.5 and Theorem 4.6. If $d \leq 2$, we may identify (46) with the morphism
$$R\Gamma_W(\mathcal{X}_s, \mathbb{Z}^c(0)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}), \mathbb{Z}[-1]) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{Wh}(\mathcal{X}_s, \mathbb{Z}), \mathbb{Z}[-1])$$
which is an equivalence of perfect complexes of abelian groups by Proposition 5.4 and [12]. □

We are now combining Proposition 6.3 and Proposition 6.6 to prove our result for the generic fiber.

Proposition 6.7. *There is an equivalence*
$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[-2d])$$
such that, for any m, there is a commutative square
$$\begin{array}{ccc} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) & \xrightarrow{\sim} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[-2d]) \\ \downarrow & & \downarrow \\ R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}/m(d)) & \xrightarrow{\sim} & R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X}_K, \mathbb{Z}/m), \mathbb{Q}/\mathbb{Z}[-2d]) \end{array}$$
where the lower horizontal map is induced by duality for usual étale cohomology of the variety $\mathcal{X}_K$.

Proof. We start with the commutative diagram:
$$\begin{array}{ccc} R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}(d)) & & \\ \uparrow & \searrow & \\ R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d)) & \longrightarrow & R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d)) \\ \downarrow & \nearrow & \\ R\Gamma_{et}(\mathcal{X}_{\mathcal{O}_{K^{un}}}, \mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d)) & & \end{array}$$
where the map
$$R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}) \otimes^L R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d)) \to R\Gamma_{et}(\mathcal{X}_{\bar{s}}, R\bar{i}^!\mathbb{Z}(d))$$
is induced by the map $\mathbb{Z} \otimes^L \mathbb{Z}(d) \to \mathbb{Z}(d)$ over $\mathcal{X}_{\mathcal{O}_{K^{un}}}$ as follows. Consider the morphism
$$\bar{i}_* R\bar{i}^!\mathbb{Z} \to \mathbb{Z} \to R\mathbf{Hom}(\mathbb{Z}(d), \mathbb{Z}(d))$$

$$\to R\mathbf{Hom}(\bar{i}_* R\bar{i}^!\mathbb{Z}(d), \mathbb{Z}(d)) \simeq \bar{i}_* R\mathbf{Hom}(R\bar{i}^!\mathbb{Z}(d), R\bar{i}^!\mathbb{Z}(d))$$

and apply $\bar{i}^*$ (here **Hom** denotes the internal Hom in the corresponding category of étale sheaves). Applying $R\Gamma(W_{\kappa(s)}, -)$ to the diagram above, we obtain the following commutative diagram in $\mathbf{D}(\mathrm{Ab})$:

$$\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) & \searrow & \\
\uparrow & & \\
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \longrightarrow & R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \\
\downarrow & & \\
R\Gamma_W(\mathcal{X}, \mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \nearrow &
\end{array}$$

Composing with the trace map $R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) \to \mathbb{Z}[-2d-1]$, we obtain the commutative diagram in $\mathbf{D}(\mathrm{Ab})$:

$$\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) & \searrow & \\
\uparrow & & \\
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \longrightarrow & \mathbb{Z}[-2d-1] \\
\downarrow & & \\
R\Gamma_W(\mathcal{X}, \mathbb{Z}) \otimes^L R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \nearrow &
\end{array}$$

It gives the following commutative diagram in $\mathbf{D}(\mathrm{Ab})$

$$\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \longrightarrow & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\downarrow & \nearrow & \\
\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) & &
\end{array}$$

We obtain the following commutative diagram

$$\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \longrightarrow & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}, \mathbb{Z}), \mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
\tau^{>a} R\Gamma_W(\mathcal{X}, \mathbb{Z}(d)) & \longrightarrow & R\mathrm{Hom}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1]) \\
 & & \downarrow \\
 & & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}), \mathbb{Z}[-2d-1])
\end{array}$$

in the derived ∞-category $\mathbf{D}^b(\mathrm{LCA})$, where the lower right map is given by Lemma 2.6. By construction of the maps (42) and (45), we obtain the following commutative diagram

$$\begin{array}{ccc}
 & \nearrow^{(45)} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
 & & \downarrow \\
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \longrightarrow & R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
\tau^{>a}R\Gamma_W(\mathcal{X},\mathbb{Z}(d)) & \longrightarrow & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
\downarrow & \nearrow_{(42)} & \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) & &
\end{array}$$

hence a commutative square

$$\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \xrightarrow{(45)} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) & \xrightarrow{(42)} & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1])
\end{array}$$

We obtain an equivalence of cofiber sequences in $\mathbf{D}^b(\mathrm{LCA})$:

$$\begin{array}{ccc}
R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}(d)) & \xrightarrow{\sim} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d)) & \xrightarrow{\sim} & R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}),\mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) & \xrightarrow{\sim} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d])
\end{array}$$

Tensoring the upper commutative square with $\mathbb{Z}/m$ gives a square equivalent to the commutative square

$$\begin{array}{ccc}
R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m(d)) & \longrightarrow & R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m),\mathbb{Q}/\mathbb{Z}[-2d-1]) \\
\downarrow & & \downarrow \\
R\Gamma_{et}(\mathcal{X},\mathbb{Z}/m(d)) & \longrightarrow & R\underline{\mathrm{Hom}}(R\Gamma_{et}(\mathcal{X}_s, Ri^!\mathbb{Z}/m),\mathbb{Q}/\mathbb{Z}[-2d-1])
\end{array}$$

where the horizontal maps are induced by the perfect pairings of Proposition 6.2. This yields the commutative square of Proposition 6.7. □

It remains to prove that

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)),\mathbb{Z}[-2d])$$

is an equivalence as well.

Lemma 6.8. *The map*

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \to R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}),\mathbb{Z})$$

is an equivalence.

Proof. We have

$$R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \simeq R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}),\mathbb{Z}),\mathbb{Z})$$

by Lemma 2.6, since $R\Gamma_{ar}(\mathcal{X},\mathbb{Z})$ is a perfect complex of abelian groups by Proposition 5.5 and Proposition 5.4. In view of the cofiber sequence

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X},\mathbb{Z}) \to R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z})$$

one is reduced to check that the map

$$R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}) \to R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}),\mathbb{Z}),\mathbb{Z})$$

is an equivalence. Recall from the proof of Proposition 6.3 that we have

$$\begin{aligned}
& R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}),\mathbb{Z}),\mathbb{Z}) \\
\simeq\ & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))[2d+1],\mathbb{Z}) \\
\simeq\ & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))[2d+1],\mathbb{R}/\mathbb{Z}[-1]) \\
\simeq\ & R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(d))^D[-2d-2] \\
\simeq\ & (\mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}))^{DD}[-2d-2] \\
\simeq\ & \mathrm{hocolim}\, R\mathrm{Hom}(R\Gamma_W(\mathcal{X},\mathbb{Z}/m(d)),\mathbb{Q}/\mathbb{Z}[-2d-1])[-1] \\
\simeq\ & R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Q}/\mathbb{Z})[-1] \\
\simeq\ & R\Gamma_W(\mathcal{X}_s, Ri^!\mathbb{Z}).
\end{aligned}$$

where the second equivalence follows from Corollary 6.5. □

Consider the pairing

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d))\underline{\otimes}^L R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \longrightarrow \mathbb{Z}[-2d] \tag{47}$$

induced by the equivalence of Proposition 6.7. Hence the induced map

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]) \tag{48}$$

is (tautologically) the equivalence of Proposition 6.7. Moreover, the map

$$R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)),\mathbb{Z}[-2d]) \tag{49}$$

induced by (47) is an equivalence as well. Indeed, applying $R\underline{\mathrm{Hom}}(-,\mathbb{Z}[-2d])$ to (48) and using Lemma 6.8, we obtain the composite equivalence

$$\begin{aligned}
R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}) &\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}),\mathbb{Z}[-2d]),\mathbb{Z}[-2d]) \\
&\xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(d)),\mathbb{Z}[-2d])
\end{aligned}$$

which is, up to equivalence, the map (49). □

6.2. Pontryagin duality. Recall that we denote by FLCA the category of locally compact abelian group of finite ranks in the sense of [13]. It follows from Proposition 5.6 that the following definition makes sense.

Definition 6.9. *Assume either $d \leq 2$ or that $(\mathcal{X}_s)^{\mathrm{red}}$ is a strict normal crossing scheme satisfying Hypothesis 5.8. For any $A \in$ FLCA and $n = 0, d$, we define*

$$R\Gamma_{ar}(\mathcal{X}_K, A(n)) := R\Gamma_{ar}(\mathcal{X}_K,\mathbb{Z}(n))\underline{\otimes}^L A;$$

$$R\Gamma_{ar}(\mathcal{X}, A(n)) := R\Gamma_{ar}(\mathcal{X},\mathbb{Z}(n))\underline{\otimes}^L A.$$

Corollary 6.10. *Assume either $d \leq 2$ or that $(\mathcal{X}_s)^{\mathrm{red}}$ is a strict normal crossing scheme satisfying Hypothesis 5.8. Then one has equivalences*

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])$$

and

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d])$$

in $\mathbf{D}^b(\mathrm{FLCA})$.

Proof. By Theorem 6.1, we have

$$\begin{array}{rcl} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) & \stackrel{\sim}{\rightarrow} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), \mathbb{Z}[-2d]) \\ & \stackrel{\sim}{\rightarrow} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}), R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]) \\ & \simeq & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z})\underline{\otimes}^L\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]) \\ & := & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}[-2d]). \end{array}$$

Applying the functor $R\underline{\mathrm{Hom}}(-, \mathbb{R}/\mathbb{Z}[-2d])$ and using Pontryagin duality, we obtain the first equivalence of the Corollary.

Similarly, we have

$$\begin{array}{rcl} R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}) & \stackrel{\sim}{\rightarrow} & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{Z}[-2d]) \\ & \simeq & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{Z}(d))\underline{\otimes}^L\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}[-2d]) \\ & := & R\underline{\mathrm{Hom}}(R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}[-2d]). \end{array}$$

□

Corollary 6.11. *Assume that $\mathcal{X}$ satisfies Hypothesis 5.8 and Hypothesis 5.7. Then for any $i \in \mathbb{Z}$, we have an isomorphism of locally compact groups*

$$H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^D$$

and an isomorphism of discrete groups

$$H^i_{ar}(\mathcal{X}_K, \mathbb{Z}) \xrightarrow{\sim} H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d))^D.$$

Proof. In view of Lemma 5.9, the equivalence in $\mathbf{D}^b(\mathrm{FLCA})$

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} R\Gamma(\mathcal{X}_K, \mathbb{Z}(d))^D[-2d]$$

induces isomorphisms

$$H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} H^i(R\Gamma(\mathcal{X}_K, \mathbb{Z}(d))^D[-2d]) \simeq H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^D$$

of locally compact abelian groups. Similarly, the equivalence in $\mathbf{D}^b(\mathrm{FLCA})$

$$R\Gamma_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \xrightarrow{\sim} R\Gamma(\mathcal{X}_K, \mathbb{Z})^D[-2d]$$

induces isomorphisms

$$H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}(d)) \xrightarrow{\sim} H^i(R\Gamma(\mathcal{X}_K, \mathbb{Z})^D[-2d]) \simeq H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z})^D$$

of compact abelian groups.

□

Remark 6.12. *It can be shown that the map $H^i_{ar}(\mathcal{X}_K, \mathbb{Z}(d)) \to H^i_{et}(\mathcal{X}_K, \widehat{\mathbb{Z}}(d))$ induces an isomorphism*

$$H^i_{ar}(\mathcal{X}_K, \mathbb{Z}(d))^{\widehat{}} \to H^i_{et}(\mathcal{X}_K, \widehat{\mathbb{Z}}(d))$$

and that the canonical map

$$H^i_{et}(\mathcal{X}_K, \mathbb{Q}/\mathbb{Z}) \to H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z})$$

is injective with dense image. Finally, the square

$$\begin{array}{ccc} H^i_{et}(\mathcal{X}_K, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \underline{\mathrm{Hom}}(H^{2d-i}_{et}(\mathcal{X}_K, \widehat{\mathbb{Z}}(d)), \mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^i_{ar}(\mathcal{X}_K, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \underline{\mathrm{Hom}}(H^{2d-i}_{ar}(\mathcal{X}_K, \mathbb{Z}(d)), \mathbb{R}/\mathbb{Z}) \end{array}$$

commutes. It follows that the lower horizontal map is uniquely determined by the upper horizontal map by continuity.

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