Maximal curves over finite fields

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**Notation and terminology**

- \( \mathcal{X} \subseteq \mathbb{P}^r(\overline{\mathbb{F}}_q) \) projective, geometrically irreducible, non-singular algebraic curve defined over \( \mathbb{F}_q \)

- \( g \) genus of \( \mathcal{X} \)
  
  If \( r = 2 \) then \( g = \frac{(d-1)(d-2)}{2} \), where \( d = \text{deg}(\mathcal{X}) \)

- \( \mathcal{X}(\mathbb{F}_q) = \mathcal{X} \cap \mathbb{P}^r(\mathbb{F}_q) \)

- \( \text{Aut}(\mathcal{X}) \) automorphism group of \( \mathcal{X} \) over \( \overline{\mathbb{F}}_q \)
Maximal curves over finite fields

Maximal curves

\( \mathcal{X} \) defined over \( \mathbb{F}_q \)

**Hasse-Weil bound**

\[ |\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + 2g\sqrt{q}. \]

**Definition**

\( \mathcal{X} \) is \( \mathbb{F}_q \)-maximal if \( |\mathcal{X}(\mathbb{F}_q)| = q + 1 + 2g\sqrt{q} \).

A necessary condition is that \( q \) is a square or \( g = 0 \)

**Example**

Hermitian curve:

\[ \mathcal{H}_\ell : Y^{\ell+1} = X^\ell + X, \quad \ell = p^h, \quad q = \ell^2 \]

\[ g = \ell(\ell - 1)/2, \quad |\mathcal{H}_\ell(\mathbb{F}_q)| = \ell^3 + 1, \quad \text{Aut}(\mathcal{H}_\ell) \cong \text{PGU}(3, \ell) \]
Problems

1. **Classification and construction**: how can we construct maximal curves?

2. **Spectrum of genera**: which values $g > 0$ occur as genera of $\mathbb{F}_q$-maximal curves for a given $q$?

3. **Applications to AG codes**: how can we compute the Weierstrass semigroup at every point of a given maximal curve?
Classification and construction of maximal curves

Coverings and Galois-coverings

\(\mathcal{X} \subseteq \mathbb{P}^r(F_q)\) and \(\mathcal{Y} \subseteq \mathbb{P}^s(F_q)\)

- If we have a non-constant \(\phi : \mathcal{X} \to \mathcal{Y}\) then \(\mathcal{Y}\) is covered by \(\mathcal{X}\) (subcover of \(\mathcal{X}\))
- \(\overline{F}_q(\mathcal{X}) : \phi^*(\overline{F}_q(\mathcal{Y}))\) is a finite field extension
- \(\overline{F}_q(\mathcal{X}) : \phi^*(\overline{F}_q(\mathcal{Y}))\) is Galois \(\to \mathcal{Y}\) is Galois-covered by \(\mathcal{X}\) (Galois-subcover of \(\mathcal{X}\))

(Kleiman-Serre, 1987)

If \(\mathcal{X}\) is \(F_q\)-maximal and \(\mathcal{Y}\) is covered by \(\mathcal{X}\) then \(\mathcal{Y}\) is \(F_q\)-maximal

Conjecture

Every \(F_q\)-maximal curve is (Galois-)covered by the Hermitian curve \(\mathcal{H}_q\)
Classification and construction of maximal curves

The conjecture is false: Natural Embedding Theorem

(Garcia-Stichtenoth, 2006)

The GS curve $X^9 - X = Y^7$ is $\mathbb{F}_{36}$-maximal and not Galois-covered by $\mathcal{H}_{33}$.

- Hermitian Variety in $\mathbb{P}^r(\mathbb{F}_q)$:
  $\mathcal{H}_{r,q} : X_2^{q+1} + X_3^{q+1} + \ldots + X_r^{q+1} = X_1^q X_0 + X_1 X_0^q$

(Korchmáros-Torres, 2001)

Un to isomorphisms, $\mathbb{F}_{q^2}$-maximal curves are

- contained in some $\mathcal{H}_{r,q}$ for some $r \geq 2$
- of degree $q + 1$
- not contained in any hyperplane of $\mathbb{P}^r(\overline{\mathbb{F}}_q)$

Definition

$r \geq 2$ is the geometrical Frobenius dimension of $\mathcal{X}$.

- If $r = 2$ then $\mathcal{X}$ is the Hermitian curve (up to isomorphism)
Classification and construction of maximal curves

The conjecture is false: The case \( r = 3 \) and the GK curve

Let \( q \) be a prime power of a prime \( p \). The GK-curve \( C : \)

\[
\begin{align*}
Z^{\frac{q^3+1}{q+1}} &= Y^{q^2} - Y, \\
X^q + X &= Y^{q+1}
\end{align*}
\]

\( \rightarrow \) Hermitian curve!

if \( \mathbb{F}_{q^6} \)-maximal. if \( q > 2 \), \( C \) is not \( \mathbb{F}_{q^6} \)-covered by \( \mathcal{H}_{q^3} \)

Question: Why are both the GK and the GS curve \( \mathbb{F}_{q^6} \)-maximal?
Classification and construction of maximal curves

The case of $\mathbb{F}_{p^2}$-maximal curves

**Conjecture, 2000**

Every $\mathbb{F}_{p^2}$-maximal curve is a subcover of the Hermitian curve $\mathcal{H}_p$

- The conjecture is true for $p \leq 5$
- No $\mathbb{F}_{p^2}$-maximal curves not Galois-covered by $\mathcal{H}_p$ are known
- Known $\mathbb{F}_{q^2}$-maximal curves have many automorphisms

**Theorem (Bartoli-M.-Torres, Adv. in Geom., 2020)**

Let $\mathcal{X}$ be an $\mathbb{F}_{p^2}$-maximal curves of genus $g$ with $p \geq 7$. If $|\text{Aut}(\mathcal{X})| > 84(g - 1)$ then $\mathcal{X}$ is Galois covered by $\mathcal{H}_p$

- Can Theorem be extended when $|\text{Aut}(\mathcal{X})| \leq 84(g - 1)$? **NO**!
- First known example: $\mathbb{F}_{71^2}$-maximal Fricke-MacBeath curve
- (Bartoli-Güneş-M., in progress) The same result is true for $\mathbb{F}_{p^4}$-maximal curves unless $\text{Aut}(\mathcal{X})$ has a very special orbits-structure
Let $q$ be a power of a prime $p$, $n \geq 3$ odd. The $\mathbb{F}_{q^{2n}}$-maximal GGS-curve is

$$C_n : \begin{cases} Z^{q^n+1}_{q+1} = Y^{q^2} - Y, \\ Y^{q+1} = X^q + X. \end{cases}$$

**Theorem (Duursma-Mak, 2012) → Conjecture for $q = 2$ (Bulletin of the Brazilian Math. Soc.)**

For $q \geq 3$ and $n \geq 5$ odd, $C_n$ is not Galois-covered by $\mathcal{H}_{q^n}$ over $\mathbb{F}_{q^{2n}}$

**Theorem (Giulietti-M.Zini, FFA, 2016) → From my Master Degree Thesis**

For $q = 2$ and $n \geq 5$ odd, $C_n$ is not Galois-covered by $\mathcal{H}_{2^n}$ over $\mathbb{F}_{2^{2n}}$

- **Key steps:** If $C_n \cong \mathcal{H}_{2^n}/G : |G| = \frac{2^n+1}{3}$ and $G$ acts semiregularly on $\mathcal{H}_{2^n}$
- **(Hartley, 1925):** Maximal subgroups of $\text{PSU}(3, 2^n)$ and their action on $\mathcal{H}_{2^n}$
- **(Dickson, 1902):** Classification of subgroups of $\text{PSL}(2, 2^{2n})$
A new infinite family of maximal curves

- (Giulietti-Korchmáros, 2009) $\text{Aut}(\mathcal{C})/\mathcal{C}_{(q^3+1)/(q+1)} \cong \text{PGU}(3, q)$ entire $\text{Aut}(\mathcal{H}_q)$

- (Guralnick-Malmskog-Pries, Güneri-Ozdemir-Stichtenoth, 2012-2013) If $n \geq 5$, $\text{Aut}(\mathcal{C}_n)/\mathcal{C}_{(q^n+1)/(q+1)} \cong \text{PGU}(3, q)_{P_{\infty}}$ maximal subgroup of $\text{Aut}(\mathcal{H}_q)$

- (Mitchell 1911, Hartley 1925) Complete list of maximal subgroups of $\text{Aut}(\mathcal{H}_q)$.

- (M.-Zini, Comm. Algebra, 2018) Let $\ell$ be a non-tangent line to $\mathcal{H}_q$. Then $\text{PGU}(3, q)_\ell \cong \text{SL}(2, q) \rtimes \mathcal{C}_{q+1}$ (maximal subgroup)

**Question**

Is it possible to construct another generalization $\{\mathcal{X}_n\}_n$ of $\mathcal{C}$ with $\mathcal{X}_3 \cong \mathcal{C}$ such that $\text{Aut}(\mathcal{X}_n)/\mathcal{C}_{(q^n+1)/(q+1)} \cong \text{PGU}(3, q)_\ell$?

Let $q$ be a power of a prime $p$, $n \geq 3$ odd,

$$\mathcal{X}_n : \begin{cases} 
Z^{q^n+1}_{q+1} = Y X^{q^2} - X \\
Y^{q+1} = X^{q+1} - 1.
\end{cases}$$
Classification and construction of maximal curves

Some observations about the family \( \{C_n\}_n, \ n \text{ odd} \)

\( \mathcal{H}_q : Y^{q+1} = X^q + X \) and \( P_\infty \) its unique point at infinity

- \( Aut(\mathcal{H}_q)_{P_\infty} = \{\alpha_{b,c} | b, c \in \mathbb{F}_q^2, \ b^{q+1} = c^q + c\} \rtimes \langle \beta_a \rangle, \)

\[ \alpha_{b,c}(X, Y) = (X + b^qY + c, Y + b), \quad \beta_a(X, Y) = (a^{q+1}X, aY), \quad \langle a \rangle = \mathbb{F}_q^* \]

\( C_n : \)

\[ \begin{align*}
Y^{q+1} &= X^q + X, \\
Z^{\frac{q^n+1}{q+1}} &= Y^{q^2} - Y.
\end{align*} \]

Let \( \alpha \in Aut(\mathcal{H}_q)_{P_\infty} \)

- If \( \alpha = \alpha_{b,c} \) then \( \alpha(Y^{q^2} - Y) = Y^{q^2} - Y \implies \) we can define \( \alpha(Z) = Z \)

- if \( \alpha = (\beta_a)^i \) then \( \alpha(Y^{q^2} - Y) = a^i(Y^{q^2} - Y) \implies \) we can define \( \alpha(Z) = \lambda_i Z \)

where \( \lambda_i^{(q^n+1)/(q+1)} = a^i \) so that

\[ \alpha \left( Z^{\frac{q^n+1}{q+1}} \right) = \lambda_i^{\frac{q^n+1}{q+1}} Z^{\frac{q^n+1}{q+1}} = a^i(Y^{q^2} - Y) = \alpha(Y^{q^2} - Y) \]
The idea behind the construction of \( \{X_n\}_n, \ n \text{ odd} \)

1. Define \( \tilde{\phi}(X,Y,Z) = (\varphi(X), \varphi(Y), -Z/(1 - \rho)X - \rho) \) then \( \tilde{\phi} \) define a birational map

\[
\mathcal{C} : \begin{cases} 
Y^{q+1} = X^q + X, \\
Z^{q_{3+1}}_{q+1} = Y^{q^2} - Y. 
\end{cases} \quad \mapsto \quad \tilde{\phi}(\mathcal{C}) := X_3 : \begin{cases} 
Y^{q+1} = X^{q+1} - 1, \\
Z^{q_{3+1}}_{q+1} = Y \frac{X^{q^2} - X}{X^{q+1} - 1}. 
\end{cases}
\]

2. Generalization (as for the GGS): \( X_n : \begin{cases} 
Y^{q+1} = X^{q+1} - 1, \\
Z^{q^n_{n+1}}_{q+1} = Y \frac{X^{q^2} - X}{X^{q+1} - 1}. 
\end{cases} \)
The idea behind the construction of \( \{X_n\}_{n, \text{ odd}} \)


1. \( X_3 \) is isomorphic to the GK curve \( C \),
2. \( X_n \) is \( \mathbb{F}_{q^{2n}} \)-maximal for every \( q \) and \( n \geq 3 \) odd,
3. For every \( n \geq 5 \) and \( q \geq 3 \) \( X_n \) is not Galois-covered by \( \mathcal{H}_{q^n} \),
4. For every \( n \geq 5 \), \( \text{Aut}(X_n)/C_{(q^n+1)/(q+1)} \cong \text{PGU}(3, q)_{\ell}, \ell \) line at infinity,
5. Even though \( g(X_n) = g(C_n) \) for every \( q \) and \( n \), the curves \( X_n \) and \( C_n \) are isomorphic if and only if \( n = 3 \)

- (Beelen-M., FFA, 2020) New other maximal curves as Galois-subcovers of \( X_n \)
- (M.- Pallozzi Lavorante, Discrete Math., 2020) Weierstrass semigroups and codes

**Questions**

- What about other maximal subgroups?
- Can we use the Natural Embedding Theorem for \( r = 4 \)?
A method to deal with the Galois covering problem

• **Question:** How can we decide in general whether a given $\mathbb{F}_{q^2}$-maximal curve is *new*, namely e.g. not Galois-covered by $\mathcal{H}_q$?

(M.-Zini, FFA, 2016)

A method to study Galois subcovers of $\mathcal{H}_q$ can be given in a purely geometric language (using Mitchell and Hartley results on $PGU(3, q)$)

• (M.-Zini, FFA, 2016) Suzuki and Ree curves $S_8$ and $R_3 \rightarrow$ proof of a conjecture by Rains and Zieve!

• (Giulietti-M.-Zini, FFA, 2016) Generalized GS curves

• (Giulietti-Kawakita-Lia-M., Advances in Geom., 2018) Wiman’s sextics

• (Bartoli-Giulietti-Kawakita-M., FFA, 2020) New curves of genus 4 and 5 constructed using Kani-Rosen Theorem

• (Bartoli-M.-Torres, Advances in Geom., 2020) Fricke-MacBeath curve

• (Giulietti-M.-Quoos-Zini, J. Number Theory, 2020) Skabelund’s curve
• (Hurwitz genus formula) If $G \leq Aut(\mathcal{H}_q) \cong PGU(3, q)$ then

$$2g(\mathcal{H}_q) - 2 = q^2 - q - 2 = 2|G|(g(\mathcal{H}_q/G) - 1) + \sum_{\sigma \in G} i(\sigma)$$

• (Mitchell 1918- Hartley 1925) The geometry and action of $G$ can be predicted according to a maximal subgroup of $PGU(3, q)$ containing it.

(M.-Zini, FFA, 2016)

The contribution $i(\sigma)$ can be computed exactly for all $\sigma \in PGU(3, q)$ depending on the order of $\sigma$ and the geometry of the given group $G$ containing it.

• IDEA: We can use the method to compute the genera of Galois-subcovers of $\mathcal{H}_q$
Open problem (Garcia-Stichtenoth-Xing, Composition Math., 2000)

Compute the genera of all Galois-subcovers of the Hermitian curve $H_q$

- **Complication:** A complete list of $G \leq PGU(3, q)$ is not known
- **General idea:** Case-by-Case analysis of subgroups of a given maximal subgroup of $PGU(3, q)$
  - $G$ fixes an $\mathbb{F}_q^2$-rational point of $H_q$ (Garcia-Stichtenoth-Xing 2000, Abdón-Quoos 2004, Bassa-Ma-Xing-Yeo 2013)
  - $G$ fixes a self-polar triangle in $\mathbb{P}^2(\mathbb{F}_q^2)$ (Giulietti-Hirschfeld-Korchmáros-Torres 2006, Dalla Volta-M. Zini, Communications in Alg., 2019)
  - $G$ fixes a $(q + 1)$-secant line to $H_q$ (M.-Zini, Communications in Alg., 2018)
  - $G$ is contained in the normalizer of a Singer cycle (Cossidente-Korchmáros-Torres 1999-2000)
  - $G$ has no fixed points or triangles (M.-Zini, J. Algebr. Comb., 2019)

(M.-Zini, J. Algebra, 2020)

Complete answer to the open problem when $q \equiv 1 \pmod{4}$
Weierstrass semigroups on points of known maximal curves

Frobenius dimension and Weierstrass semigroups

Let \( \mathcal{X} \) be a curve and let \( P \in \mathcal{X} \)

**Definition: Weierstrass semigroup at \( P \)**

\[
H(P) = \{ \rho \in \mathbb{Z}_{\geq 0} \mid \text{there exists a rat. func. } f \text{ with } (f)_{\infty} = \rho P \}
\]

**Weierstrass gap Theorem**

\[
G(P) = \mathbb{N}_0 \setminus H(P) \text{ contains exactly } g(\mathcal{X}) \text{ elements called gaps}
\]

**Theorem**

If \( \mathcal{X} \) is \( \mathbb{F}_{q^2} \)-maximal and \( P \in \mathcal{X}(\mathbb{F}_{q^2}) \) then \( q, q + 1 \in H(P) \) and \( r \leq \text{number of non-gaps less than } q + 1 \)

- The structure of \( H(P) \) is almost always the same: Weierstrass points
- Independent interest (e.g. Stöhr-Voloch Theory) and main ingredient to construct AG codes!
- Hermitian curve: \( r = 2 \rightarrow \text{(Garcia-Viana, 1986)} \)
- GK curve: \( r = 3 \) (smallest possible) \( \rightarrow ? \)
Weierstrass semigroups on points of known maximal curves

Weierstrass semigroups on the GK curve $C$

- (Giulietti-Korchmáros, 2009) $H(P) = \langle q^3 - q^2 + q, q^3, q^3 + 1 \rangle$, $P \in C(\mathbb{F}_{q^2})$

- (Fanali-Giulietti, 2010) $H(P)$, $P \in C(\mathbb{F}_{q^6}) \setminus C(\mathbb{F}_{q^2})$ and $q \leq 3$

- (Duursma, 2011) $H(P)$, $P \in C(\mathbb{F}_{q^6}) \setminus C(\mathbb{F}_{q^2})$ and $q \leq 9$

**Conjecture (Duursma 2011, IEEE Trans. Inf. Theory)**

Let $P \in C(\mathbb{F}_{q^6}) \setminus C(\mathbb{F}_{q^2})$. Then

$$H(P) = \langle q^3 - q + 1, q^3 + 1, q^3 + i(q^4 - q^3 - q^2 + q - 1) \mid i = 0, \ldots, q - 1 \rangle$$

- Nothing is known for $P \not\in C(\mathbb{F}_{q^6})$

**Main questions:**

1. Which points of $C$ are Weierstrass points?
2. Is it possible to determine $H(P)$ for all $P \in C$? (in particular, to prove the conjecture)?
Weierstrass semigroups on points of known maximal curves

Weierstrass semigroups on the GK curve $C$

(Theorem, Beelen-M., FFA, 2018)

Let $P \in C$. Then

- $H(P) = \langle q^3 - q^2 + q, q^3, q^3 + 1 \rangle$, if $P \in C(\mathbb{F}_{q^2})$;
- $H(P) = \langle q^3 - q + 1, q^3 + 1, q^3 + i(q^4 - q^3 - q^2 + q - 1) \mid i = 0, \ldots, q - 1 \rangle$, if $P \in C(\mathbb{F}_{q^6}) \setminus C(\mathbb{F}_{q^2}) \rightarrow$ proof of Duursma’s conjecture;
- $H(P) = \mathbb{N} \setminus G$, if $P \notin C(\mathbb{F}_{q^6})$, where
  
  $G = \{ iq^3 + kq + m(q^2 + 1) + \sum_{s=1}^{q-2} n_s((s+1)q^2) + j + 1 \mid i, j, k, m, n_1, \ldots, n_{q-2} \in \mathbb{Z}_{\geq 0}, j \leq q - 1 \text{ and } i + j + k + mq + \sum_s n_s((s+1)q - s) \leq q^2 - 2 \}$.

(Corollary, Beelen-M., FFA, 2018)

The set of Weierstrass points of the GK curve is $W = C(\mathbb{F}_{q^6})$

→ One of the very few curves in which all Weierstrass semigroups and points are known!
Weierstrass semigroups on points of known maximal curves

Why is it a conjecture? Main difficulties

Let \( P = P_{(a,b,c)} \in \mathcal{C}(\mathbb{F}_{q^6}) \setminus \mathcal{C}(\mathbb{F}_{q^2}) \) and

\[
T := \langle q^3 - q + 1, q^3 + 1, q^3 + i(q^4 - q^3 - q^2 + q - 1) \mid i = 0, \ldots, q - 1 \rangle
\]

• Two aims
  1. \( T \subseteq H(P) \)
  2. \( |\mathbb{N} \setminus T| = g(C) \)

• We need functions \( f_\rho \in \mathbb{F}_{q^6}(C) \) with \( (f_\rho)_\infty = \rho P, \rho \) generator of \( T \)

• An explicit description of \( f_\rho(x, y, z) \) can be really complicated

• Example \( q = 3 \): \( f_{q^4-q^2+q-1} = f_{74} \) is

\[
(2x^3b^{72} + 2x^3b^{64} + 2x^3b^{56} + 2x^3b^{48} + 2x^3b^{40} + 2x^3b^{32} + 2x^3b^{24} + 2x^3b^{16} + 2x^3b^{8} + 2x^3 + 2x^2yb^{27} + x^2yb^3 + x^2b^{36} + 2x^2b^{12} + xy^2b^{54} + 2xy^2b^6 + xy^3b^{27} + 2xy^3b^3 + xyb^{63} + 2xyb^{39} + 2xa^3b^{36} + xa^3b^{12} + xb^{72} + xb^{48} + xb^{24} + y^3b^{73} + y^3b^{65} + y^3b^{49} + y^3b^{41} + 2y^3b^{33} + y^3b^{25} + y^3b^{17} + 2y^3b^9 + y^3b^{54} + 2y^2a^3b^6 + 2y^2b^{66} + y^2b^{18} + 2ya^6b^{27} + ya^6b^3 + ya^3b^{63} + 2a^9b^{72} + 2a^9b^{64} + 2a^9b^{56} + 2a^9b^{48} + 2a^9b^{40} + 2a^9b^{32} + 2a^9b^{24} + 2a^9b^{16} + 2a^9b^8 + 2a^9 + a^6b^{36} + 2a^6b^{12} + a^3b^{72} + a^3b^{48} + 2ya^3b^{39} + 2yb^{75} + 2yb^{51} + 2yb^{27} + a^3b^{24} + b^{84} + 2b^{12})/( -a^{27} - x + b^{27}y + c^{27}z )^3
\]
Weierstrass semigroups on points of known maximal curves

Our proof of the Conjecture: \( T \subseteq H(P) \)

- There exists \( \tilde{z}_P \) with \( (\tilde{z}_P) = (q^3 + 1)P - (q^3 + 1)P_\infty \) (Natural Emb. Theorem)
- \( \tilde{x}_P = -a^q - x + b^q y \) (tangent line at \( P|Q \) on the Hermitian curve)
- \( P \notin C(\mathbb{F}_{q^2}) \): \( k = 1, 2 \), \( k \)-Frobenius twist of \( \tilde{x}_P \): \( \tilde{x}_P^{(k)} = -a^{q^{2k+1}} - x + b^{q^{2k+1}} y \)
Weierstrass semigroups on points of known maximal curves

Our proof of the Conjecture: \( T \subseteq H(P) \)

(Lemma, Beelen-M., FFA, 2018)

Let

\[
 f_i = \frac{\tilde{x}^{q_i} \cdot \tilde{x}^{(2)}_P}{(\tilde{x}^{(1)}_P)^i \cdot \tilde{z}^{q-i+1}_P}, \quad i = 1, \ldots, q - 1.
\]

Then

\[
 (f_i)_{\infty} = q^3 + i(q^4 - q^3 - q^2 + q - 1)P
\]

while \( 1/\tilde{z}_P, (y - b)/\tilde{z}_P \) and \( \tilde{x}_P/\tilde{z}_P \) give \( \langle q^3 - q + 1, q^3, q^3 + 1 \rangle \subseteq H(P) \). So

\( T \subseteq H(P) \)

- (M.- Pallozzi Lavorante, Discrete Math., 2020) \( H(P) \) where \( P \in X_n(\mathbb{F}_{q^2}) \)
- (Bartoli-M.-Zini, Acta Arith., 2020) \( H(P) \) at every \( P \) and \( W \): Suzuki curve \( S_q \)
- (Beelen-Landi-M., 2020) \( H(P) \) at every \( P \) and \( W \): Skabelund curve
Thank you

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