Faltings height and Néron-Tate height of a theta divisor

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based on joint work with Farbod Shokrieh

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Hence $\mathrm{h}(A) = 2g \cdot \mathrm{h}'_L(\Theta)$.

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Our goal: $\mathrm{h}_F(A) =$
$2g \cdot \mathrm{h}'_L(\Theta) + d \cdot (\text{a sum of local factors indexed by the places of } k)$.

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- note $I(\mathbf{A}, \lambda) > 0$.

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What about the general case?

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- the real torus $\Sigma = \mathrm{Hom}(X, \mathbb{R})/Y$ is a principally polarized tropical abelian variety canonically associated to $(\mathbf{A}, \lambda)$

A principally polarized tropical abelian variety is a real torus $\Sigma = \mathrm{Hom}(X, \mathbb{R})/Y$ where Y, X is a pair of finitely generated free abelian groups together with an isomorphism $\phi\colon Y \xrightarrow{\sim} X$ and a bilinear map $b\colon Y \times X \to \mathbb{Z}$ such that $b(\cdot, \phi(\cdot))$ is positive definite.

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We have the Voronoi polytope with center the origin

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Here μ_L denotes the Lebesgue measure on V, normalized to give $\mathrm{Vor}(0)$ unit volume.

Alternatively, consider the tropical Riemann theta function

$$\Psi(\nu) := \min_{u' \in Y} \left\{ \frac{1}{2} \|u'\|^2 + [\nu, u'] \right\} \tag{1}$$

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on V is Y-invariant and hence descends to Σ. Explicitly

$$\|\Psi\|(\nu) = \frac{1}{2} \min_{u' \in Y} \|\nu + u'\|^2$$

for all $\nu \in V$. The tropical moment of Σ can alternatively be written as

$$2 \int_{\Sigma} \|\Psi\| \, \mathrm{d}\, \mu_H \,,$$

where μ_H is the Haar measure on Σ, normalized to give Σ unit volume. It is a non-negative rational number, zero iff $\Sigma = (0)$.

General case

When $(\mathbf{A}, \lambda)$ is a principally polarized abelian variety over F as above we denote by $I(\mathbf{A}, \lambda)$ the tropical moment of the associated principally polarized tropical abelian variety Σ.

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$$\mathrm{h}_F(A) = 2g\,\mathrm{h}'_L(\Theta) - \kappa_0\, g + \frac{1}{d}\left(\sum_{v \in M(k)_0} I(A_v, \lambda_v) \log Nv \right. \\ \left. + 2 \sum_{v \in M(k)_\infty} I(A_v, \lambda_v) \right) .$$

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$$\mathrm{h}_F(A) \geq -\kappa_0\, g + \frac{2}{d}\sum_{v\in M(k)_\infty} I(A_v,\lambda_v)\,, \quad \mathrm{h}_F(A) > -\kappa_0\, g\,,$$

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previously obtained by Bost (1996).

Side remark: in the function field setting with semistable reduction we obtain

$$\mathrm{h}(A) = 2g\,\mathrm{h}'_L(\Theta) + \sum_{v \in S_0} I(A_v, \lambda_v)$$

with S_0 the set of closed points of S.

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$$\mathrm{h}(A) = 2g\,\mathrm{h}'_L(\Theta) + \sum_{v \in S_0} I(A_v, \lambda_v)$$

with S_0 the set of closed points of S. This refines the well-known fact (Moret-Bailly, Szpiro, Faltings-Chai) that $\mathrm{h}(A) \geq 0$.

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$$12\,d\,\mathrm{h}_F(A) = \sum_{v\in M(k)_0} \mathrm{ord}_v\,\Delta_v \log Nv - \sum_{v \in M(k)_\infty} \log\left((2\pi)^{12}|\Delta(\tau)|(\mathrm{Im}\,\tau)^6\right),$$

which is well known (Faltings, Silverman).

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Then $\tau(\Gamma)$ is independent of the choice of q.

Let G be a model of Γ, and fix an orientation on G. We then have a natural boundary map $\partial\colon C_1(G,\mathbb{Z}) \to C_0(G,\mathbb{Z})$ with kernel $H_1(G,\mathbb{Z})$.

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Let $X = Y$ and $b\colon Y \times X \to \mathbb{Z}$ the restriction of $[\cdot,\cdot]$ to $Y \times Y$. We obtain a principally polarized tropical abelian variety $\Sigma = \mathrm{Hom}(X,\mathbb{R})/Y$ from Γ called the tropical jacobian of Γ.

Theorem (-, Shokrieh): *Let $I(\Gamma)$ denote the tropical moment of the tropical jacobian of Γ. Then the relation*

$$I(\Gamma) = \frac{1}{8}\ell(\Gamma) - \frac{1}{2}\tau(\Gamma)$$

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Side remark: this allows fast computation of $I(\Gamma)$, starting from the discrete Laplacian on a model G. Only need to perform a couple of matrix multiplications and Gauss eliminations where the matrices involved have size $|V(G)|$. Computation of tropical moment of general lattices is expected by (some) experts to be NP-hard.

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<u>Corollary:</u> *Let C be a smooth projective geometrically connected curve of genus g with semistable reduction over k. Let (A, λ) be its jacobian. Then the formula*

$$\mathrm{h}_F(A) = 2g\,\mathrm{h}'_L(\Theta) - \kappa_0\, g + \frac{1}{d}\left(\sum_{v \in M(k)_0} \left(\frac{1}{8}\ell(\Gamma_v) - \frac{1}{2}\tau(\Gamma_v) \right) \log Nv + 2 \sum_{v \in M(k)_\infty} I(A_v, \lambda_v) \right)$$

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and thus one should have

$$I(\Gamma_n) = \frac{n+1}{8} - \frac{1}{2}\left(\frac{1}{4(n+1)} + \frac{1}{12}\frac{n^2}{n+1}\right) = \frac{n}{12} + \frac{n}{6(n+1)}\,.$$

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The lattice $H_1(\Gamma_n, \mathbb{Z})$ is isometric with the root lattice A_n. Conway-Sloane in their book compute that

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Let $(\pi\colon \mathcal{A} \to S, \mathcal{L}')$ be a so-called Moret-Bailly model of (A, L') over S. Then $\overline{\pi_* \mathcal{L}'}$ with the ℓ^2-metric derived from the canonical admissible metric on L' is a hermitian line bundle on S, and the formula

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For the tautological Moret-Bailly model we calculate $\widehat{\deg}\,\overline{\pi_*\mathcal{L}'}$ explicitly.