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Effective sup-norm bounds on average for cusp forms of even weight.

1. Introduction. Before giving the main results, we want to remind of the existing qualitative results in this direction. We need the following notation.

- $\mathbb{H} = \{z \in \mathbb{C} \mid z = x + iy, y > 0\}$, the upper half-plane.
- $\Gamma \subset PSL_2(\mathbb{R})$, Fuchsian subgroup of the first kind.
- $M = \Gamma \backslash \mathbb{H}$, the corresponding orbifold.
- g_Γ = genus of M.

For $k \in \mathbb{N}_{>0}$, we then define

- S_{2k}^{Γ} = space of cusp forms of weight $2k$ for Γ
- $d_{2k} = \dim_{\mathbb{C}}(S_{2k}^{\Gamma})$

The crucial quantity to be considered (already some years ago and revisited in Ma's talk) is

$$S^{\Gamma}_{2k}(z) := \sum_{j=1}^{d_{2k}} |f_j(z)|^2 \cdot y^{2k},$$

where $\{f_1, \dots, f_{d_{2k}}\}$ is an orthonormal basis of S^{Γ}_{2k} (with respect to the Petersson inner product).

We are then interested in (optimal) sup-norm bounds for the quantity $S^{\Gamma}_{2k}(z)$.

We have (as recalled yesterday)

Γ cocompact: $\sup_{z \in \mathbb{H}} S^{\Gamma}_{2k}(z) = O_{\Gamma}(k)$

Γ cofinite: $\sup_{z \in \mathbb{H}} S^{\Gamma}_{2k}(z) = O_{\Gamma}(k^{3/2})$

(and there is also uniformity in covers)

Today's aim is to make the implied constants effective (maybe not optimal at this point in time, but this can be done later ...)

2. Results. We start by giving our quantitative estimates in the cocompact setting.

In addition, we need that ℓ_Γ denotes the (hyperbolic) length of the shortest closed geodesic on M.

Theorem A. With the above notation, we have

$$\frac{2k-1}{4\pi} \leq \sup_{z\in\mathbb{H}} S_{2k}^{\Gamma}(z) \leq \frac{2k-1}{4\pi} + C_\Gamma e^{-\delta_\Gamma k}$$

where

$$C_\Gamma = \frac{3e^{12\pi g_\Gamma/\ell_\Gamma}}{\pi(g_\Gamma-1)} \frac{(\cosh(\ell_\Gamma)+1)^2}{\log((\cosh(\ell_\Gamma)+1)/2)},$$

$$\delta_\Gamma = \frac{1}{2}\log\left(\frac{\cosh(\ell_\Gamma)+1}{2}\right).$$

Before being able to state the main result in the cofinite setting, we need some more notation.

- $\mathcal{F}$ denotes a fixed fundamental domain for Γ.
- $\mathcal{P} = \{p_1, \dots, p_e\}$ with $p_1 = i\infty$ denotes the sets of cusps of $\mathcal{F}$.
- $\sigma_j \in PSL_2(\mathbb{R})$ is the scaling matrix for p_j and that $\sigma_j^{-1} \Gamma_{p_j} \sigma_j \cong \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$.

- For $Y > 1$, define $\mathcal{F}_j^Y \subset \mathcal{F}$ as $\sigma_j^{-1}(\mathcal{F}_j^Y) = \{ z = x+iy \in \mathbb{H} \mid -1/2 \le x \le 1/2,\ y \ge Y \}$
- $\mathcal{F}_Y = cl(\mathcal{F} \setminus (\mathcal{F}_1^Y \cup \dots \cup \mathcal{F}_e^Y))$ ($\mathcal{F}_Y = \mathcal{F}$, if Γ is cocompact).

- $\mathcal{E} = \{e_1, \dots, e_n\}$ denotes the set of elliptic fixed points of $\mathcal{F}$
- n_j = order of e_j, $\Theta_j = 2\pi / n_j$, and $\Theta_\Gamma = \min_{j=1,\dots,n} \Theta_j$

Then we have

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Theorem B. With the above notation, we have in the cofinite setting for $k \geq 2$:

(i) For $Y > 1$, there exist effectively computable constants B_Y and σ_Y such that

$$\sup_{z \in \mathcal{F}_Y} S_{2k}^{\Gamma}(z) \leq \frac{2k-1}{4\pi}\Big(1 + 6\sum_{\mathfrak{c}_j \in \mathcal{E}} (n_j - 1)\Big) + 12(2k-1)\, B_Y\, \sigma_Y^{-(k-2)},$$

(ii) For $Y_0 \geq 8/\sqrt{15}$, $k \geq 4\pi Y_0$, $Y = 2Y_0$, there exists an effectively computable constant B_{k, Y_0} such that

$$\sup_{z \in \mathcal{F}_j^Y} S_{2k}^{\Gamma}(z) \leq \frac{2k-1}{4\pi} + \frac{3(2k-1)}{2\pi} \times \times \left(B_{k,Y_0} + \frac{\sqrt{k}\, e^{5/4}}{\sqrt{\pi}}\right).$$

(iii) For $Y_0 \geq 8/\sqrt{15}$, $2 \leq k \leq 4\pi Y_0$, $Y \geq 2Y_0$, the same bound as in (i) holds for $\sup_{z \in \mathcal{F}_j^Y} S_{2k}^{\Gamma}(z)$.

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Remark. Explanation for the quantities B_Y, σ_Y, and B_{ε,Y_0}.

1.) We have

$$B_Y = \frac{e^{3\,\mathrm{diam}(\mathcal{F}_Y)/2}}{\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}_Y)}.$$

2.) Recall (for $z, w \in \mathbb{H}$)

$$\sigma(z,w) = \cosh^2\left(\frac{\mathrm{dist}_{\mathrm{hyp}}(z,w)}{2}\right).$$

Then, we have

$$\sigma_Y := \inf_{\substack{z \in \mathcal{F}_Y \\ \gamma \in \Gamma \setminus \Gamma_\varepsilon}} \sigma(z, \gamma z)$$

$(\Gamma_\varepsilon = \Gamma_{e_1} \cup \ldots \cup \Gamma_{e_n})$.

3.) $B_{\varepsilon,Y_0} := 2\pi\, Y_0^{-4}\, B_{Y_0}\, 4^{-\varepsilon+3} \left(\dfrac{\varepsilon}{2\pi}\right)^4.$

In order to effectively bound σ_φ, we note introducing

a) $0 < m_\varphi < M_\varphi$ such that
$m_\varphi \leq \operatorname{Im}(\sigma_j^{-1} z) \leq M_\varphi$
for all $z \in \mathcal{F}_\varphi$ (and all j)

b) Assume $\partial\mathcal{F} = \bigcup \mathcal{S}$ ($\mathcal{S}$ = line segments) and $\mathcal{f} = \{S\}$.
Define

$$\mu_\Gamma := \inf_{\substack{S \in \mathcal{f} \\ e \in \mathcal{E} \setminus S}} \operatorname{dist}_{hyp}(S, e)$$

Then, we have

$$\sigma_\varphi \geq \min\left\{ \frac{\cosh(\ell_\Gamma)+1}{2},\ \sinh^2(\mu_\Gamma)\sin^2(\theta_\Gamma/2)+1,\ \frac{m_\varphi^2}{4}+1,\ \frac{1}{4M_\varphi^2}+1 \right\} \geq 1.$$

Example $\Gamma = PSL_2(\mathbb{Z})$ and $Y = 16/\sqrt{15}$.

$$\sup_{z \in \mathbb{H}} S_{2k}^{\Gamma}(z) \leq \begin{cases} \dfrac{31(2k-1)}{4\pi} + 1090\,(2k-1)\,1.014^{-(k-2)} & (k \geq 2,\ z \in \mathcal{F}_Y) \\[2ex] \dfrac{31(2k-1)}{4\pi} + 1090\,(2k-1)\,1.014^{-(k-2)} & (2 \leq k \leq 25,\ z \in \mathcal{F}_1^Y) \\[2ex] \dfrac{(2k-1)}{4\pi} + \dfrac{9(2k-1)}{2\pi}\sqrt{k} & (k \geq 26,\ z \in \mathcal{F}_1^Y) \end{cases}$$

3. Strategy of proof. We start from the Laplacian

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+2iky\frac{\partial}{\partial x},$$

acting on

$$\mathcal{H}_k^\Gamma := \left\{ \varphi : \mathbb{H} \to \mathbb{C} \;\middle|\; \varphi(\gamma z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^k \varphi(z) \text{ and } \|\varphi\| < \infty \right\}.$$

We then observe that

$$S_{2k}^\Gamma = \ker\left(\Delta_k - k(1-k)\right).$$

Then, we consider the resolvent kernel for Δ_k, denoted $G_k(s;z,w)$, well-defined for

$$s \in W_k = \mathbb{C} \setminus \{k-n, -k-n \mid n \in \mathbb{N}\}.$$

and

$$g_k(s;z,w) = G_k(s;z,w) - G_k(s+1;z,w).$$

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Looking at the spectral expansion of $g_k(s;z,w)$, we get with $\lambda = s(s-1)$ and $\mu = t(t-1)$ with $t = s+1$

$$\sum_{j=0}^{\infty} \left(\frac{1}{\lambda_j - \lambda} - \frac{1}{\lambda_j - \mu} \right) |\varphi_j(z)|^2 + \text{Eisenstein-part}$$

$$= -\frac{1}{4\pi} \left(\psi(s+k) + \psi(s-k) - \psi(t+k) - \psi(t-k) \right)$$

$$+ \sum_{\substack{\gamma \in \Gamma \\ \gamma \neq id}} \left(\frac{cz+d}{c\bar{z}+d} \right)^k \left(\frac{\gamma z - \bar{z}}{z - \gamma \bar{z}} \right)^k g_k(s;z,\gamma z)$$

We then let $s = k + \varepsilon$ $(\varepsilon > 0)$, $t = s+1 = k+1+\varepsilon$ and restrict to the $(\lambda_j = 0)$-part (which brings in the ONB $\{f_j\}_{j=1}^{d_{2k}}$)

By neglecting all the other positive terms on the right-hand side, we get the upper bound:

$$S^{\Gamma}_{2k}(z) \leq \frac{(2k-1+\varepsilon)(1+\varepsilon)}{4\pi} +$$

$$\frac{\varepsilon(2k+\varepsilon)(2k-1+\varepsilon)(1+\varepsilon)}{2(k+\varepsilon)} \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq id}} |g_k(k+\varepsilon; z, \gamma z)|$$

- Observing now that

$$\sum_{\substack{\gamma\in\Gamma \\ \gamma\neq id}} |g_k(k+\varepsilon; z, \gamma z)| \leq \frac{3}{2\pi\varepsilon} \sigma(z, \gamma z)^{-(k+\varepsilon)}$$

the upper bound for $z \in \mathcal{F}_Y$ follows by let $\varepsilon \to 0$.

- The bounds for the cuspidal neighborhoods have to be treated separately.