On the slopes of the lattice of modular forms

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1. Measures associated to successive maxima.

Let $\overline{E} = (E, (||\cdot||_v)_v)$ be an adelic vector bundle of rank $r = \mathrm{rank}(\overline{\mathrm{E}})$ over $\mathbb{Q}$.
For every place v of $\mathbb{Q}$ the vector space $E \otimes \mathbb{C}_v$ is thus equipped with a norm $||s||_v$.

The successive maxima $(\lambda_i)_{i=1}^r = (\lambda_i(\overline{E}))_{i=1}^r$ of $\overline{E}$ are defined as follows. The real number $\lambda_i(\overline{E})$ is the largest real number λ such that the set of elements $s \in E$ satisfying

$$\lambda(s) := -\sum_v \log ||s||_v \geq \lambda,$$

generates a $\mathbb{Q}$-vector space of dimension at least i.

Let now X be a projective variety of dimension d over $\mathbb{Q}$, and let L be an ample line bundle on X, endowed with a continuous adelic metric $(|\cdot|_{L,v})_v$, in the sense of Zhang. In particular, we assume that for all but a finite number of places, the metrics $(|\cdot|_{L,v})_v$ come from a single integral model of (X, L) over $\mathbb{Z}$.

The $\mathbb{Q}$-vector space $H^0(X, L^{\otimes n})$ is then an adelic vector bundle, in the sense above, if equipped with the family of norms

$$||s||_{L^{\otimes n},v} = \sup_{x \in X(\mathbb{C}_v)} |s(x)|_{L^{\otimes n},v}.$$

Let r_n be the dimension of $H^0(X, L^{\otimes n})$ and $\lambda_{i,n}, i = 1 \cdots r_n$, the successive maxima of $\overline{H^0(X, L^{\otimes n})}$.

Theorem 1(Chen)
The sequence of probability measures

$$\nu_n = \frac{1}{r_n} \sum_{i=1}^{r_n} \delta_{\frac{1}{n}\lambda_{i,n}}$$

converges weakly to a probability measure ν with compact supports.

2. Modular forms and Petersson norms.

We first recall some work of Bost and Kuhn concerning the interpretation of holomorphic modular forms of weight $12k$ for $\mathrm{SL}_2(\mathbb{Z})$ as sections of the k^{th} power for $k \geq 1$ of a particular metrized line bundle on $\mathbb{P}^1_{\mathbb{Z}}$.

Let $\mathbb{H}$ be the upper half plane and let $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ be the modular group. Then $X = \Gamma \backslash (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$ has a natural structure as a Riemann surface. The classical j function of $z \in \mathbb{H}$ has expansion

$$j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} a_n q^n \quad \text{in} \quad q = e^{2\pi i z}.$$

The map $z \to j(z)$ defines an isomorphism $X \to \mathbb{P}^1_{\mathbb{C}}$.

The volume form of the hyperbolic metric on $\mathbb{H}$ is

$$\mu = \frac{dx \wedge dy}{y^2} = \frac{i}{2}\frac{dz \wedge d\overline{z}}{\mathrm{Im}(z)^2}.$$

This form has singularities at the cusp and at the elliptic fixed points of Γ.

Define

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q + \sum_{n>1} b_n q^n$$

to be the normalized cusp form of weight 12 for Γ. Let $S_{i\infty}$ be the unique cusp of X.

Suppose k is a positive integer. We define the line bundle

$$\mathcal{M}_{12k}(\Gamma)_\infty = O_X(S_{i\infty})^{\otimes k}$$

to be the line bundle of modular forms of weight $12k$ with respect to Γ. This is shown to be compatible with the usual classical definition of modular forms.

In particular, there is an isomorphism

$$M_{12k}(\Gamma) \to H^0(X, O_X(S_{i\infty})^{\otimes k})$$

between the space $M_{12k}(\Gamma)$ of classical modular forms $f = f(z)$ of weight $12k$ and $H^0(X, O_X(S_{i\infty})^{\otimes k})$, which sends f to the element $\frac{f}{\Delta^k}$ in the function field $\mathbb{C}(X) = \mathbb{C}(j)$.

The Petersson metric $|\ |_\infty$ on $\mathcal{M}_{12k}(\Gamma)_\infty$ is defined by

$$|f|_\infty^2(z) = |f(z)|^2(4\pi\,\mathrm{Im}(z))^{12k}$$

if f is a meromorphic section of $\mathcal{M}_{12k}(\Gamma)_\infty$.

It can be shown that this metric is logarithmically singular with respect to the cusp and the elliptic fixed points of X.

Following Kuhn, we define an integral model of X to be

$$\mathcal{X} = \mathrm{Proj}(\mathbb{Z}[Z_0, Z_1])$$

with Z_0 and Z_1 corresponding to the global sections $j \cdot \Delta$ and Δ of the ample line bundle $\mathcal{M}_{12}(\Gamma)_\infty$.

The point $S_{i\infty}$ defines a section $\overline{S}_{i\infty}$ of $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{Z}$.
We extend $\mathcal{M}_{12k}(\Gamma)_\infty$ to the line bundle

$$\mathcal{M}_{12k}(\Gamma) = O_{\mathcal{X}}(\overline{S}_{i\infty})^{\otimes k}$$

on $\mathcal{X}$.

The model $\mathcal{X}$ then gives natural metrics $|\ |_v$ at all non-archimedean places v for the induced line bundle $\mathcal{M}_{12k}(\Gamma)_{\mathbb{Q}}$ on the general fiber $X_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} X_{\mathbb{Z}}$. When v is the infinite place of $\mathbb{Q}$, we let $|\ |_v$ be the Petersson metric $|\ |_{\infty}$.

Theorem 2

Let $\{\lambda_{i,12k}\}_{i=1}^{k}$ be the successive maxima associated to $\mathcal{S}_{2k}(\Gamma)$ with respect to the L^2 Hermitian norm defined by the Petersson metric.

i. The sequence of probability measures

$$\nu_{12k} = \frac{1}{k} \sum_{i=1}^{k} \delta_{\frac{1}{k}\lambda_{i,12k}}$$

converges weakly as $k \to \infty$ to a probability measure ν.

ii. The support of the measure ν is bounded above by $2\pi + 6(1 - \log(12)) = -2.62625\ldots$. The support of ν is not bounded below.

Furthermore, as $k \to \infty$, the proportion of successive maxima which are produced by normalized Hecke eigen cusp forms in $\mathcal{S}_{12k}(\Gamma)$ goes to 0.