Bergman kernels on punctured Riemann surfaces

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Jean-Michel Bismut

- Thesis (1973) : backward stochastic differential equations
- Malliavin Calculus : 1977-1983
- Local index theory : 1983–
- Geometric hypoelliptic Laplacians : 2002–

Bergman kernel on complex manifolds

- Dolbeault cohomology
- Kodaira map
- Bergman kernel

Our point of view : spectral gap + localization

Punctured Riemann surfaces

- Bergman kernel on complete manifolds
- Punctured Riemann surfaces
- Applications : Cusp forms

Dolbeault complex

- X compact complex manifold, $n = \dim X$.
- E a holomorphic vector bundle on X.
- $\overline{\partial}^E : \Omega^{0,q}(X,E) := \mathscr{C}^\infty(X, \Lambda^q(T^{*(0,1)}X) \otimes E) \to \Omega^{0,q+1}(X,E)$ the Dolbeault operator :

$$\overline{\partial}^E \Big(\sum_j \alpha_j \xi_j \Big) = \sum_j (\overline{\partial}\alpha_j)\xi_j .$$

ξ_j local hol. frame of E, and $\alpha_j \in \Omega^{0,q}(X)$.

$$(\overline{\partial}^E)^2 = 0.$$

Dolbeault cohomology

- Dolbeault cohomology of X with values in E :

$$H^q(X,E) := H^{(0,q)}(X,E) := \frac{\ker(\overline{\partial}^E|_{\Omega^{0,q}})}{\mathrm{Im}(\overline{\partial}^E|_{\Omega^{0,q-1}})}.$$

  finite dimensional !
  Measure the obstruction to solve the equation $\bar{\partial} g = f$.
- $H^0(X,E)$ space of holomorphic sections of E on X.

Kodaira embedding

- L positive line bundle/compact complex manifold X.
- Kodaira embedding theorem (1954) : $\exists p_0$, s.t. for $p \geqslant p_0$, Kodaira map

$$\Phi_p : X \longrightarrow \mathbb{P}(H^0(X, L^p)^*),$$
$$\Phi_p(x) = \{s \in H^0(X, L^p) : s(x) = 0\},$$

  is well-defined, and is a holomorphic embedding.
- + Chow theorem, this implies X is algebraic variety : $\exists$ homogenous polynomials $\{f_j\}_j$ on $z \in \mathbb{C}^N$ s.t. $X = \{[z] \in \mathbb{CP}^{N-1} : f_j(z) = 0 \ \forall j\}$.

Metric aspect of Kodaira map

- (L, h^L) positive hol. Herm. line bundle on X
- Kodaira map $\Phi_p : X \longrightarrow \mathbb{P}(H^0(X, L^p)^*)$. We have $\Phi_p^* \mathscr{O}(1) \simeq L^p$, and

$$h^{\Phi_p^* \mathscr{O}(1)}(x) = P_p(x, x)^{-1} h^{L^p}(x).$$

$P_p(x, x) \in \mathscr{C}^\infty(X)$ Bergman kernel on the diagonal.

Bergman kernel

- (E, h^E) hol. Herm. vector bundle on X. $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ Kähler form. $dv_X = \frac{\omega^n}{n!}$ Riem. vol. form on X
- L^2-metric on $H^0(X, L^p \otimes E)$
$$\langle s, s' \rangle = \int_X \langle s, s' \rangle (x) dv_X(x).$$
- $P_p : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$ orth. proj. Bergman kernel $P_p(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)_{x'}$ smooth kernel of P_p.
- $\{s_j\}$ orth. basis of $H^0(X, L^p \otimes E)$, then
$$P_p(x, x') = \sum_j s_j(x) \otimes s_j(x')^*.$$
If $E = \mathbb{C}$, $P_p(x, x) = \sum_j |s_j(x)|^2$.

Asymptotic expansion

- Take $E = \mathbb{C}$. $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ Kähler form.
- $\exists\, b_r(x) \in \mathscr{C}^\infty(X)$, $b_0 = 1, \quad b_1 = \frac{1}{8\pi} r^X$.

$$\left| P_p(x,x) - \sum_{r=0}^{k} b_r(x) p^{n-r} \right|_{\mathscr{C}^l} \leqslant C_{k,l} p^{n-k-1}.$$

- Initial by Tian, established by Catlin, Zelditch (1998) by using parametrix of Boutet de Monvel-Sjöstrand (1976) for Bergman kernel on the disc bundle $\Omega = \{z \in L^* : |z|_{h^L} \leq 1\}$.
- Corollary (Tian 1990) : Set of Fubini-Study forms is dense in space of Kähler forms in Kähler class $c_1(L)$.

$$\left| \frac{1}{p}\, \Phi_p^*(\omega_{FS}) - \omega \right|_{\mathscr{C}^l(X)} \leqslant C_l/p.$$

More applications

- Lu compute some coefficients b_r by using peak section method (L^2-method) in complex geometry.
- Donaldson : existence of constant scalar curvature Kähler metric $\omega \in c_1(L)$ relates to Mumford-Chow stability of X.

Spectral gap property : our starting point

- $D_p := \sqrt{2}\left(\overline{\partial}^{L^p} + \overline{\partial}^{L^p,*}\right)$. (Dirac operator !)
- Hodge + Kodaira : for $p \gg 0$, $H^0(X, L^p) = \operatorname{Ker} D_p$.
- Spectral gap property : for $p \gg 0$,

$$\mathrm{Spec}(\mathrm{D}_\mathrm{p}^2) \subset \{0\} \cup [4\pi \mathrm{p} - \mathrm{C}_\mathrm{L}, +\infty[.$$

Bismut-Vasserot : complex case
Ma, Marinescu : symplectic case

Dai-Liu-Ma, Ma-Marinescu : Idea of the proof

- $\mathrm{Spec}(\mathrm{D}_\mathrm{p}^2) \subset \{0\} \cup [4\pi \mathrm{p} - \mathrm{C_L}, +\infty[\implies p \gg 0,$

$$P_p = e^{-tD_p^2} - e^{-tD_p^2}\, 1_{[2\pi p, \infty[}(D_p^2).$$

- When $p \to \infty$, $P_p \sim e^{-tD_p^2}$. Use heat kernel $e^{-tD_p^2}$.
- Principal : • spectral gap $\Longrightarrow$ the problem is local
  • Analytic localization technique of Bismut-Lebeau in local index theory $\Longrightarrow$ Asymptotic expansion and the effective way to compute the coefficients.
- Dai-Liu-Ma : Asymptotic expansion for $P_p(x, x')$, works for ∘ orbifold ∘ symplectic

Toeplitz operator I

- $P_p : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)$ orthogonal projection. Bergman projection !
- Berezin-Toeplitz quantization of $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$:

$$T_{f,p} = P_p f P_p \in \mathrm{End}(H^0(X, L^p \otimes E)).$$

- A Toeplitz operator is a family of operators $\{T_p \in \mathrm{End}(H^0(X, L^p \otimes E))\}_{p\in\mathbb{N}^*}$ s. t. $\exists g_l \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ s.t. $\forall k \in \mathbb{N}, p \in \mathbb{N}^*$,

$$\left\| T_p - \sum_{l=0}^{k} p^{-l} T_{g_l,p} \right\| \leqslant C_k\, p^{-k-1}.$$

Berezin (1970), Boutet de Monvel-Guillemin (1981), Bordemann-Meinrenken-Schlichenmaier, Ma-Marinescu

Geometric Quantization (Kostant, Souriau)

- Classical phase space :(X, ω)
  Quantum phase space $H^0(X, L)$
- Classical observables : Poisson algebra $\mathscr{C}^\infty(X)$,
  Quantum observables : linear operators on $H^0(X, L)$
- Semi-classical limit : $H^0(X, L^p)$, $p \to \infty$ is a way to relate the classical and quantum observables.

Toeplitz operator II

- Ma-Marinescu (2008, 2012) : $\forall f, g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, $T_{f,p}\, T_{g,p}$ is a Toeplitz operator, and
$$T_{f,p}\, T_{g,p} = T_{fg,p} + T_{-\frac{1}{2\pi}\langle \nabla^{1,0} f, \overline{\partial}^E g\rangle_\omega, p}\, p^{-1} + \mathcal{O}(p^{-2}).$$
Roughly, our character : $\{T_p \in \mathrm{End}(H^0(X, L^p \otimes E))\}$ is Toeplitz operator iff it has the same type off-diagonal asymptotics expansion as Bergman kernel $P_p(x, x')$. Thus Toeplitz operators form an algebra.
- It's useful in our recent study with Jean-Michel Bismut, Weiping Zhang on the asymptotics of the analytic torsion for flat vector bundles.
- When $E = \mathbb{C}$, in the Kähler case, B-M-S (1994) : $T_{f,p}\, T_{g,p} = T_{fg,p} + \mathcal{O}(p^{-1})$.

Deformation quantization

- Ma-Marinescu (2008) : Symplectic case, $C_0(f,g) = fg$

$$T_{f,p}\,T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),\,p} + \mathcal{O}(p^{-\infty}).$$

- For $E = \mathbb{C}$, Berezin-Toeplitz $*$-product :

$$f *_\hbar g := \sum_{l=0}^{\infty} \hbar^l C_l(f,g) \in \mathscr{C}^\infty(X)[[\hbar]] \text{ for } f,g \in \mathscr{C}^\infty(X).$$

$\Longrightarrow$ geometric canonical, associative $*$-product $f *_\hbar g$.

$$f *_\hbar g - g *_\hbar f = \sqrt{-1}\{f,g\}\hbar + \mathcal{O}(\hbar^2).$$

- Existence of formal $*$-product :
  on symplectic manifolds by De Wilde, Lecomte (1983).
  on Poisson manifolds by Kontsevich (1996).

Bergman kernel on complete manifolds

- (X, ω_X) complete Kähler manifold, $\dim X = n$, (L, h) Herm. hol. line bundle on X.
- Theorem (Ma-Marinescu 2007) : Assume $\exists\, \varepsilon, C > 0$ s.t.

$$iR^L \geq \varepsilon \omega_X, \quad \mathrm{Ric}_{\omega_X} \geq -C\omega_M$$

Then $\exists\, \boldsymbol{b}_j \in \mathscr{C}^\infty(M)$ s.t. $\forall$ compact set $K \subset X$, $k, m \in \mathbb{N}$, $\exists\, C > 0$ s.t. for $p \in \mathbb{N}^*$,

$$\Big\| \frac{1}{p^n} B_p(x) - \sum_{j=0}^{k} \boldsymbol{b}_j(x) p^{-j} \Big\|_{\mathscr{C}^m(K)} \leqslant C\, p^{-k-1},$$

$$\boldsymbol{b}_0 = \frac{c_1(L,h)^n}{\omega_X^n}, \;\; \boldsymbol{b}_1 = \frac{\boldsymbol{b}_0}{8\pi}\,(r_\omega - 2\Delta_\omega \log \boldsymbol{b}_0),$$

r_ω, Δ_ω scalar curvature, Laplacian w.r.t. $\omega := c_1(L,h)$.

Punctured Riemann surfaces $\Sigma = \overline{\Sigma} \smallsetminus D$

- $\overline{\Sigma}$ compact Riemann surface, $D = \{a_1, \ldots, a_N\} \subset \overline{\Sigma}$ finite set. $\Sigma = \overline{\Sigma} \smallsetminus D$
- L hol. line bundle on $\overline{\Sigma}$, h singular metric on L s.t.
  - (α) h smooth over Σ, $\exists$ a trivialization of L on $\overline{V_j} \ni a_j$ s.t. $|1|_h^2(z_j) = \big|\log(|z_j|^2)\big|$, $\forall j$.
  - (β) $\exists\, \varepsilon > 0$ s.t. the (smooth) curvature R^L of h satisfies

$$iR^L \geq \varepsilon\omega_\Sigma \text{ over } \Sigma \text{ and } iR^L = \omega_\Sigma \text{ on } V_j := \overline{V_j} \smallsetminus \{a_j\}.$$

- Poincaré metric on punctured unit disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$

$$\omega_{\mathbb{D}^*} := \frac{idz \wedge d\overline{z}}{|z|^2 \log^2(|z|^2)} \,.$$

- $\Longrightarrow \omega_\Sigma = \omega_{\mathbb{D}^*}$ on V_j and (Σ, ω_Σ) is complete.

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$$H^0_{(2)}(\Sigma, L^p) = \left\{ S \in H^0(\Sigma, L^p) : \|S\|^2_{\boldsymbol{L}^2} := \int_\Sigma |S|^2_{h^p}\, \omega_\Sigma < \infty \right\}$$

- We have

$$H^0_{(2)}(\Sigma, L^p) \subset H^0\big(\overline{\Sigma}, L^p\big).$$

- Bergman kernel function : $\{S^p_\ell\}_{\ell=1}^{d_p}$ an orthonormal basis of $H^0_{(2)}(\Sigma, L^p)$, then

$$B_p(x) = \sum_{\ell=1}^{d_p} |S^p_\ell(x)|^2_{h^p} \ : \Sigma \to \mathbb{R}.$$

$B^{\mathbb{D}^*}_p$ w.r.t. $\big(\mathbb{D}^*, \omega_{\mathbb{D}^*}, \mathbb{C}, \big|\log(|z|^2)\big|^p \big| \quad \big|\big)$.

- Theorem (Auvray-Ma-Marinescu 2016) : Assume that $(\Sigma, \omega_\Sigma, L, h)$ fulfill conditions (α) and (β). Let $a \in D$, and $0 < r < e^{-1}$ as above. Then $\forall k \in \mathbb{N}, \ell > 0$, $\alpha \geq 0$, $\exists\, C$ s.t. for $p \gg 1$, on $\mathbb{D}^*_{r/2} \times \mathbb{D}^*_{r/2}$, we have

$$\Big| B_p^{\mathbb{D}^*}(x,y) - B_p(x,y) \Big|_{C^k(h^p)} \leq C p^{-\ell} \Big| \log(|x|^2) \Big|^{-\alpha} \Big| \log(|y|^2) \Big|^{-\alpha}.$$

- Theorem (Auvray-Ma-Marinescu 2016) : Assume that $(\Sigma, \omega_\Sigma, L, h)$ fulfill conditions (α) and (β). Then $\forall \ell, m \in \mathbb{N}$, and $\delta > 0$, $\exists\, C > 0$ s.t. $\forall p \in \mathbb{N}^*$, and $z \in V_1 \cup \ldots \cup V_N$

$$\Big| B_p - B_p^{\mathbb{D}^*} \Big|_{\mathscr{C}^m}(z_j) \leq C p^{-\ell} \, \Big| \log(|z_j|^2) \Big|^{-\delta}.$$

- Corollary : As $p \to \infty$,

$$\sup_{x\in\Sigma} B_p(x) = \sup_{x\in\Sigma, 0\neq\sigma\in H^0_{(2)}(\Sigma,L^p)} \frac{|\sigma(x)|^2_{h^p}}{\|\sigma\|^2_{\boldsymbol{L}^2}} = \Big(\frac{p}{2\pi}\Big)^{3/2} + \mathcal{O}(p).$$

- For $p \geq 2$, the set

$$\Big\{ \Big(\frac{\ell^{p-1}}{2\pi(p-2)!}\Big)^{1/2} z^\ell : \ell \in \mathbb{N},\, \ell \geq 1 \Big\}$$

forms an orthonormal basis of $H^p_{(2)}(\mathbb{D}^*)$. Thus $B^{\mathbb{D}^*}_p(z) = \frac{\left|\log(|z|^2)\right|^p}{2\pi(p-2)!} \sum_{\ell=1}^\infty \ell^{p-1}|z|^{2\ell}$.

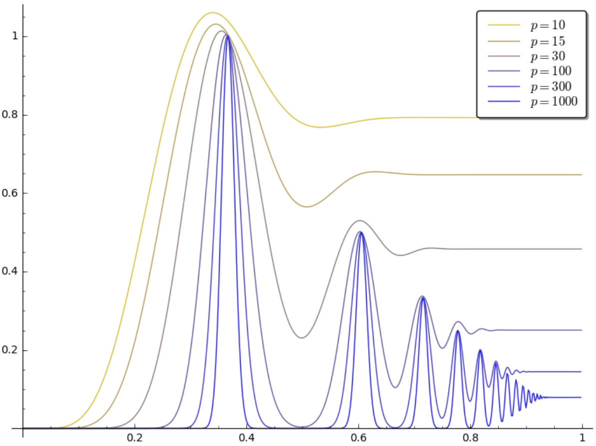

FIGURE – Functions $\left(\frac{2\pi}{p}\right)^{3/2} \frac{\left|\log(x^p)\right|^{p+1}}{2\pi(p-1)!} \sum_{\ell=1}^{\infty} \ell^p x^{p\ell}$ on $(0,1)$

Geometric description

- $\overline{\Sigma}$ compact Riemann surface of genus g
  $D = \{a_1, \dots, a_N\} \subset \overline{\Sigma}$.
- The following conditions are equivalent :
  (i) $\Sigma = \overline{\Sigma} \smallsetminus D$ admits a complete Kähler-Einstein metric ω_Σ with $\mathrm{Ric}_{\omega_\Sigma} = -\omega_\Sigma$,
  (ii) $2g - 2 + N > 0$,
  (iii) the universal cover of Σ is the upper-half plane $\mathbb{H}$,
  (iv) $L = K_{\overline{\Sigma}} \otimes \mathscr{O}_{\overline{\Sigma}}(D)$ is ample.
  (v) $\Sigma := \Gamma \backslash \mathbb{H}$, $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ a geometrically finite Fuchsian group of the first kind, without elliptic elements.
- The Kähler-Einstein metric ω_Σ is induced by the Poincaré metric on $\mathbb{H}$.

- $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\} \subset \mathbb{C}$ upper-half plane.

$$\mathrm{PSL}(2,\mathbb{R}) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) : \det\gamma = 1\right\}/\pm 1$$

acting on $\mathbb{C}$ as

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Poincaré metric on $\mathbb{H}$

$$\omega_{\mathbb{H}} = \frac{i dz \wedge d\overline{z}}{4y^2}.$$

- $\mathcal{S}_{2p}^{\Gamma}$ space of *cusp forms* of weight $2p$ of Γ :

$$f \in \mathscr{O}(\mathbb{H}) : f(\gamma z) = (cz+d)^{2p} f(z),\ z \in \mathbb{H}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and its limit at any cusp of Γ is zero.

- Mumford (1977)

$$\Phi : f \in \mathcal{S}_{2p}^{\Gamma} \to f dz^{\otimes p} \in H^0(\mathbb{H}, K_{\mathbb{H}}^p)$$

induces an isomorphism

$$\Phi : \mathcal{S}_{2p}^{\Gamma} \to H^0\big(\overline{\Sigma}, L^p \otimes \mathscr{O}_{\overline{\Sigma}}(D)^{-1}\big) \cong H^0_{(2)}\big(\Sigma, L^p\big).$$

- Petersson scalar product on $\mathcal{S}_{2p}^{\Gamma}$

$$\langle f, g\rangle := \int_{\text{fund. domain of } \Gamma} f(z)\overline{g(z)}(2y)^{2p}\,\frac{1}{2}y^{-2}dxdy.$$

- Φ is an isometry !

- $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ a geometrically finite Fuchsian group of the first kind without elliptic elements.
  B_p^Γ Bergman kernel function of cusp forms of weight $2p$
- Theorem (AMM) • If Γ is cocompact, as $p \to +\infty$

$$B_p^\Gamma(x) = \frac{p}{\pi} + \mathcal{O}(1), \qquad \text{uniformly on } \Gamma\backslash\mathbb{H}.$$

  • If Γ is not cocompact then

$$\sup_{x\in\Gamma\backslash\mathbb{H}} B_p^\Gamma(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}(p), \qquad \text{as } p \to +\infty.$$

- Let $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R})$ be a fixed Fuchsian subgroup of the first kind without elliptic elements and let $\Gamma \subset \Gamma_0$ be any subgroup of finite index.
- Theorem (AMM) • If Γ_0 is cocompact, then

$$B_p^{\Gamma}(x) = \frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \qquad \text{as } p \to +\infty.$$

• If Γ_0 is not cocompact then

$$\sup_{x \in \Gamma \backslash \mathbb{H}} B_p^{\Gamma}(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}_{\Gamma_0}(p), \qquad \text{as } p \to +\infty.$$

and constants in $\mathcal{O}_{\Gamma_0}(1)$, $\mathcal{O}_{\Gamma_0}(p)$ depend solely on Γ_0.

- $\Gamma_0 \subset \mathrm{PSL}(2,\mathbb{R})$ a fixed Fuchsian subgroup of the first kind. $\{x_j\}_{j=1}^q$ orbifold points of $\Gamma_0\backslash\mathbb{H}$.
- $\Gamma \subset \Gamma_0$ subgroup of finite index, $\pi_\Gamma : \Gamma\backslash\mathbb{H} \to \Gamma_0\backslash\mathbb{H}$ projection.
- Theorem (AMM) • If Γ_0 is cocompact, then as $p \to +\infty$

$$B_p^\Gamma(x) = \frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1), \quad \text{uniformly on } (\Gamma\backslash\mathbb{H}) \smallsetminus \bigcup_{j=1}^q \pi_\Gamma^{-1}(U_{x_j}).$$

On each $\pi_\Gamma^{-1}(U_{x_j})$ we have as $p \to +\infty$,

$$B_p^\Gamma(x) = \left(1 + \sum_{\gamma \in \Gamma_{x_j^\Gamma} \smallsetminus \{1\}} \exp\left(ip\theta_\gamma - p(1-e^{i\theta_\gamma})|z|^2\right)\right)\frac{p}{\pi} + \mathcal{O}_{\Gamma_0}(1).$$

- • If Γ_0 is not cocompact then as $p \to \infty$

$$\sup_{x \in \Gamma \backslash \mathbb{H}} B_p^{\Gamma}(x) = \left(\frac{p}{\pi}\right)^{3/2} + \mathcal{O}_{\Gamma_0}(p).$$

  constants in $\mathcal{O}_{\Gamma_0}(1)$, $\mathcal{O}_{\Gamma_0}(p)$ depend solely on Γ_0.
- Abbes and Ullmo (1995), Michel and Ullmo (1998)
- Theorem (Friedman, Jorgenson and Kramer (2013)) :

$$\sup_{x \in \Gamma \backslash \mathbb{H}} B_p^{\Gamma}(x) = \begin{cases} \mathcal{O}_{\Gamma_0}(p) & \text{if } \Gamma_0 \text{ is cocompact,} \\ \mathcal{O}_{\Gamma_0}(p^{3/2}) & \text{if } \Gamma_0 \text{ is not cocompact.} \end{cases}$$

Idea of the proof

- Kodaira Laplacian

$$\square_p = \overline{\partial}^{L^p}\overline{\partial}^{L^p*} + \overline{\partial}^{L^p*}\overline{\partial}^{L^p} : \Omega^{(0,\bullet)}(\Sigma, L^p) \to \Omega^{(0,\bullet)}(\Sigma, L^p).$$

  Spectral gap : $\mathrm{Spec}(\square_p) \subset \{0\} \cup [Cp, \infty) \Longrightarrow$ The problem is local !

- Weighted elliptic estimates and Weighted Sobolev inequalities :

$$\|f\|_{C^0(\Sigma,\omega_\Sigma)} \leq c_0 \|f\|_{\boldsymbol{L}^{1,3}_{\mathrm{wtd}}}.$$

  with

$$\|f\|_{\boldsymbol{L}^{1,k}_{\mathrm{wtd}}} := \int_\Sigma \rho\big(|f| + \ldots + |(\nabla^\Sigma)^k f|_{\omega_\Sigma}\big)\omega_\Sigma.$$

  for $\rho \in \mathscr{C}^\infty(\Sigma, [1, +\infty))$, $\rho = \big|\log(|z_j|^2)\big|$ near $a_j \in D$.

Idea of the proof

- for $\gamma > \frac{1}{2}$, $\ell \in \mathbb{N}^*$, $\exists C > 0$ s.t. $\forall x, y \in \mathbb{D}^*_{r/2}$,

$$\left|B_p^{\mathbb{D}^*} - B_p^{\Sigma}\right|_{C^0}(x,y) \leq C p^{-\ell} \left|\log(|x|^2)\right|^{\gamma} \left|\log(|y|^2)\right|^{\gamma}.$$

- Use the observation

$$H^0_{(2)}(\Sigma, L^p) = \{\sigma \in H^0(\overline{\Sigma}, L^p), \sigma|_D = 0\},$$

to conclude : $\forall \delta > 0$, $\ell \in \mathbb{N}^*$, $\exists C > 0$ s.t. $\forall x, y \in \mathbb{D}^*_{r/2}$,

$$\left|B_p^{\mathbb{D}^*} - B_p^{\Sigma}\right|_{C^0}(x,y) \leq C p^{-\ell} \left|\log(|x|^2)\right|^{-\delta} \left|\log(|y|^2)\right|^{-\delta}.$$

Thank you !