Continuity of plurisubharmonic envelopes in non-archimedean geometry and test ideals

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joint work with
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- Boucksom, S.; Favre, C.; Jonsson, M.: **Solution to a non-Archimedean Monge-Ampère equation.** J. Amer. Math. Soc. 28 (2015), no. 3, 617–667.
- Boucksom, S.; Favre, C.; Jonsson, M.: **Singular semipositive metrics in non-Archimedean geometry.** J. Algebraic Geom. 25 (2016), no. 1, 77–139.
- Burgos Gil, J.; Gubler, W.; Jell, P.; Künnemann, K.; Martin, F.: **Differentiability of non-archimedean volumes and non-archimedean Monge-Ampère equations** (with an appendix by R. Lazarsfeld). `arXiv:1608.01919`
- Gubler, W.; Jell,P.; Künnemann, K.; Martin, F: Continuity of **Plurisubharmonic Envelopes in Non-Archimedean Geometry and Test Ideals** (with an Appendix by J. Burgos Gil and M. Sombra). `arXiv:1712.00980`

Table of contents

The archimedean Calabi-Yau problem

- (M, ω) Kähler manifold of dimension n,
- ω a real, closed, positive $(1,1)$-form in M, in holomorphic coordinates $(z_1, \ldots, z_n) : U \to \mathbb{C}^n$ on M

$$\omega|_U = \frac{-1}{2\pi i} \sum_{k,l=1}^{n} h_{kl} dz_k \wedge d\overline{z}_l$$

where the hermitian matrix $(h_{kl})_{kl}$ is positive definite.

Theorem (Calabi (uniqueness 1957), Yau (existence 1978)
Given a smooth positive volume form Ω with $\int_M \Omega = \int_M \omega^{\wedge n}$ there exists a unique real smooth closed $(1,1)$-form α on M with $[\alpha] = [\omega]$ in $H^2_{DR}(M)$ and

$$\alpha^{\wedge n} = \Omega \quad \text{(Monge-Ampère equation)}.$$

- Let L holom. line bundle on M with $c_1(L) = [\omega] \in H^2_{DR}(M)$.
- Hodge theory yields a smooth hermitian metric $\| \ \|$ on L (unique up to scaling) such that the curvature form satisfies $c_1(L, \| \ \|) = \alpha$.

Get equality of measures

$$c_1(L, \| \ \|)^{\wedge n} = \mu \quad \text{(Monge-Ampère equation)}$$

where Radon measure μ is defined by Ω and Monge-Ampère measure $c_1(L, \| \ \|)^{\wedge n}$ is given by

$$f \mapsto \int_M f \cdot c_1(L, \| \ \|)^{\wedge n}$$

with local density on holomorphic chart (U, z) of M

$$\frac{1}{(2\pi i)^n} \det\left(\frac{\partial^2 \log \|s\|^2}{\partial z_k \partial \bar{z}_l}\right) dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n.$$

Bedford-Taylor Theory. If metric $\| \ \|$ on L is semipositive in the sense that it can be approximated uniformly by semipositive smooth metrics $(\| \ \|_i)_{i\in\mathbb{N}}$ on L then measure $c_1(L, \| \ \|)^{\wedge n}$ can defined as weak limit of the measures $c_1(L, \| \ \|_i)^{\wedge n}$.

Theorem (Kołodziej 1998) If measure μ on M has (locally) L^p-density with respect to Lebesgue measure for some $p > 1$ then there exist a semipositive continuous metric $\| \ \|$ on L such that

$$c_1(L, \| \ \|)^{\wedge n} = \mu.$$

The non-archimedean Calabi-Yau problem

Setup for the rest of this talk:

- $(K, |\ |)$ non-archimedean complete discretely valued field discrete valuation ring $K^\circ = \{x \in K \mid |x| \leq 1\}$ (noetherian), maximal ideal $K^{\circ\circ} = \{x \in K \mid |x| < 1\}$, residue class field $\tilde{K} = K^\circ / K^{\circ\circ}$,
- examples $(\mathbb{C}((T)), \mathbb{C}[[T]], \mathbb{C})$ and $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}/p\mathbb{Z})$ for prime p,
- X normal projective variety over K of dimension n,
- $X^{\mathrm{an}} :=$ Berkovich analytification of X. Consists of pairs $(p, |\ \ |_p)$ where $p \in X$ and $|\ \ |_p$ is absolute value on $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$ which extends $|\ \ |$ on K. Equip X^{an} with coarsest topology such that $\pi\colon X^{\mathrm{an}} \longrightarrow X,\ (p, |\ \ |_p) \mapsto p$ continuous and for all U open in X and $f \in \mathcal{O}_X(U)$

$$|f| : U^{\mathrm{an}} = \pi^{-1}(U) \to \mathbb{R},\ (p, |\ \ |_p) \mapsto |f(p)| := |f + \mathfrak{m}_{X,p}|_p$$

is continuous as well.

Metrics. Let L be a line bundle on X. A continuous metric $\| \ \|$ on L associates with every $s \in \Gamma(U, L)$ for $U \subseteq X$ open a continuous function $\|s\| \colon U^{\mathrm{an}} \to \mathbb{R}_{\geq 0}$ such that $\|f \cdot s\| = |f| \, \|s\|$ for all $f \in \mathcal{O}_X(U)$ and $\|s\| > 0$ if s is a frame of L.

Models. A model $\mathscr{X}$ of X over K° is a normal proper flat scheme over $S := \operatorname{Spec} K^\circ$ with generic fibre X.

Reduction. Let $\mathscr{X}$ be a model of X over K°. There is a canonical reduction map

$$\mathrm{red} \colon X^{\mathrm{an}} \longrightarrow \mathscr{X}_s$$

where $\mathscr{X}_s := \mathscr{X} \otimes_{K^\circ} \tilde{K}$ denotes the special fibre of $\mathscr{X}$.

Shilov Points. A generic point η_V of an irreducible component V of $\mathscr{X}_s$ has a unique preimage x_V in X^{an} (Berkovich).

Model metrics. Let $\mathscr{X}$ be a model of X over K° and $\mathscr{L}$ a line bundle on $\mathscr{X}$ with $\mathscr{L}|_X = L$. Pair $(\mathscr{X}, \mathscr{L})$ determines a unique continuous metric $\| \ \|_{\mathcal{L}}$ on L such that a frame s of $\mathscr{L}$ over $\mathscr{U} \subseteq \mathscr{X}$ open satisfies $\|s\|_{\mathscr{L}} = 1$ on $\mathrm{red}^{-1}(\mathscr{U}_s)$. Such a metric is called an algebraic metric. A continuous metric $\| \ \|$ on L is a model metric if some power $(L^{\otimes k}, \| \ \|^{\otimes k})$ is an algebraic metric.

Nef line bundles. A line bundle $\mathscr{L}$ on model $\mathscr{X}$ is called nef if

$$\deg_{\tilde{K}}(\mathscr{L}|_C) \geq 0$$

for any proper curve C in $\mathscr{X}_s$.

Semipositive model metrics. A model metric $\| \ \|$ on L is called semipositive if some power is induced by a nef line bundle.

Semipositive metrics. A uniform limit of semipositive model metrics on L is called semipositive metric.

Chambert-Loir measures. There is unique way to associate with semipositive metrized line bundles $(L_1, \| \ \|_1), \ldots, (L_n, \| \ \|_n)$ a (positive) Radon measure $c_1(L_1, \| \ \|_1) \wedge \ldots \wedge c_1(L_n, \| \ \|_n)$ on X^{an} of total mass $\deg_X(c_1(L_1) \ldots c_1(L_n))$ which is multilinear and continuous in the $(L_i, \| \ \|_i)$, satisfies a projection formula, and is given as follows for model metrics.

Chambert-Loir measures for model metrics. Let $\mathscr{X}$ be a model of X. For line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_n$ on $\mathscr{X}$ models of L on X the Chambert-Loir measure $c_1(L, \| \ \|_{\mathscr{L}_1}) \wedge \ldots \wedge c_1(L, \| \ \|_{\mathscr{L}_n})$ is the discrete signed measure

$$\sum_{V \in \mathscr{X}_s^{(0)}} \ell_{\mathcal{O}_{\mathscr{X}_s,V}}(\mathcal{O}_{\mathscr{X}_s,V}) \deg_V(\mathscr{L}_1 \cdots \mathscr{L}_n) \, \delta_{x_V}$$

on X^{an}, where $x_V \in X^{\mathrm{an}}$ is Shilov point determined by $\mathrm{red}(x_V) = \eta_V$ and δ_{x_V} is the Dirac measure in x_V.

Assume in the following that the projective variety X is smooth.

SNC models. An SNC model $\mathscr{X}$ of X is a regular model such that $(\mathscr{X}_s)_{\mathrm{red}}$ is simple normal crossing divisor in $\mathscr{X}$.

Skeleta. Each SNC model determines a skeleton $\Delta_{\mathscr{X}} \subseteq X^{\mathrm{an}}$ (Berkovich, BFJ). The skeleton is geometric realization of a simplicial complex whose 0-dimensional vertices are given by the Shilov points associated with irreducible components of $\mathscr{X}_s$.

Fact. If char $\tilde{K} = 0$ then $K \cong \tilde{K}((T))$ and SNC models exist.

Varieties over K of geometric origin. We say that X is of geometric origin if there exists a normal curve B over a field k and a closed point b on B such that K° is the completion of the discrete valuation ring $\mathcal{O}_{B,p}$ and X is defined over the function field $\kappa(B)$ of B.

Theorem (Boucksom, Favre, Jonsson 2015). Fix L ample line bundle on X and μ positive Radon measure on X^{an} of total mass $\deg c_1(L)^n$. If

(i) $\mathrm{char}(\tilde{K}) = 0$,

(ii) X is of geometric origin,

(iii) μ is supported on skeleton of some SNC model of X

then there exists semipositive metric $\| \ \|$ on L such that

$$c_1(L, \| \ \|)^{\wedge n} = \mu.$$

Remarks.

- Thuillier (2005) X curve without (i)-(ii)
- Y. Liu (2011) X totally degenerate abelian variety w.o. (i)-(ii)
- Yuan, Zhang (2016) uniqueness up to scaling without (i)-(iii)
- Thm. BFJ without (ii) (Burgos, Gubler, Jell, K., Martin 2016)

Strategy (Boucksom, Favre, Jonsson)

- Fix semipositive reference metric $\| \ \|_0$ on L^{an},
- consider energy $E(\| \ \|, \| \ \|_0)$ defined as

$$\frac{-1}{n+1}\sum_{j=0}^{n}\int_{X^{\mathrm{an}}}\log\frac{\| \ \|}{\| \ \|_0}c_1(L,\| \ \|)^{\wedge j}\wedge c_1(L,\| \ \|_0)^{n-j}$$

and maximize

$$\| \ \| \longmapsto E(\| \ \|, \| \ \|_0) - \int_{X^{\mathrm{an}}} -\log\frac{\| \ \|}{\| \ \|_0}d\mu,$$

- use orthogonality principle + differentiability of $E \circ P$ where

$$P(\| \ \|) = \inf_{\text{pointwise}}\{\| \ \|' \mid \| \ \|' \text{ semipos. model metric} \| \ \| \leq \| \ \|'\}$$

denotes the semipositive envelope of the metric,

Strategy (Boucksom, Favre, Jonsson)

- fix continuous metric $\| \ \|$ on L^{an},
- use Assumption (i) and the theory of multiplier ideals to show continuity of the semipositive envelope

$$P(\| \|) = \inf_{\text{pointwise}} \{ \| \|' \mid \| \|' \text{ semipos. model metric} \| \| \leq \| \|' \},$$

- use Assumption (ii) that X is of geometric origin to show:

Orthogonality Principle. If the line bundle L is ample then

$$\int_{X^{\mathrm{an}}} \log \frac{P(\| \ \|)}{\| \ \|} c_1(L, P(\| \ \|))^{\wedge n} = 0,$$

i.e. the support of the Chambert-Loir measure is orthogonal to the locus where the metric differs from its envelope.

Strategy (Burgos, Gubler, Jell, K., Martin 2016).

Definition. For $\| \ \|_1, \| \ \|_2$ on L define non-archimedean volume

$$\operatorname{vol}(L, \| \ \|_1, \| \ \|_2) = \limsup_{m\to\infty} \frac{n!}{m^{n+1}} \operatorname{length}_{K^\circ}\left(\frac{\widehat{H}^0(X, L^{\otimes m}, \| \ \|_1^{\otimes m})}{\widehat{H}^0(X, L^{\otimes m}, \| \ \|_2^{\otimes m})}\right).$$

Remark. Relate vol to energy E + show $\operatorname{vol}(L, P(\| \ \|), \| \ \|) = 0$.

Theorem (Burgos, Gubler, Jell, K., Martin) If $\| \ \|$ continuous semipositive metric on L and $f\colon X^{\mathrm{an}} \to \mathbb{R}$ continuous then $t \in \mathbb{R} \mapsto \operatorname{vol}(L, \| \ \| e^{-tf}, \| \ \|)$ is differentiable at $t = 0$ and

$$\frac{d}{dt}\Big|_{t=0} \operatorname{vol}(L, \| \ \| e^{-tf}, \| \ \|) = \int_{X^{\mathrm{an}}} f\, c_1(L, \| \ \|)^{\wedge n}.$$

Remark. Theorem gives orthogonolity principle without assumption X of geometric origin if $P(\| \ \|)$ is continuous.

Results in positive characteristic

Assume for the rest of this talk that

- $\operatorname{char} K = p > 0$,
- X a smooth projective surface over K,
- L an ample line bundle on X.

Theorem (Gubler, Jell, K. ,Martin). Let X be of geometric origin from perfect ground field k. If $\| \ \|$ is continuous metric on L^{an} then $P(\| \ \|)$ is continuous semipositive metric on L^{an}.

Corollary (Gubler, Jell, K. ,Martin). In situation of Theorem let μ be a positive Radon measure on X^{an} with $\mu(X^{\mathrm{an}}) = \deg c_1(L)^2$ supported on skeleton of projective SNC-model of X . Then the non-archimedean Monge-Ampère equation $c_1(L, \| \ \|)^{\wedge 2} = \mu$ has continuous semipositive solution $\| \ \|$.

Remarks.

- Continuity of envelopes holds for curves in any characteristic (Gubler, Jell, K., Martin).
- Proof of our Theorem uses results about asymptotic test ideals (Mustaţă) and resolution of singularities in positive characteristic (Cossart–Piltant).
- Under the assumption of resolution of singularities in positive characteristic our results generalize from surfaces X to varieties X of arbitrary dimension.
- Our results generalize furthermore to varieties coming by base change in codimension one from families over higher dimensional varieties.

Asymptotic test ideals

- Introduced Hochster, Huneke (1990) theory of tight closure,
- Y smooth variety over a perfect field k with $p := \mathrm{char}\, k > 0$,
- $F : Y \to Y$ Frobenius, i.e. $F|_{|X|} = \mathrm{id}_{|X|}$ and $F^*(s) = s^p$,
- $\omega_{Y/k} = \det \Omega^1_{Y/k} = \mathcal{O}_Y(K_{Y/k})$ for canonical divisor $K_{Y/k}$,
- fix $e \in \mathbb{Z}$ and ideal (sheaf) $\mathfrak{a}$ of $\mathcal{O}_Y$,
- there exists unique ideal $\mathfrak{a}^{[p^e]}$ of $\mathcal{O}_Y$ with

$$\mathfrak{a}^{[p^e]}(U) = \langle u^{p^e} \mid u \in \mathfrak{a}(U) \rangle_{\mathrm{i}deal} \quad (U \text{ open affine in } Y),$$

- Cartier isomorphism yields trace map $\mathrm{Tr}\colon F_*(\omega_{Y/k}) \to \omega_{Y/k}$,
- (Def. Mustaţă) there exists unique ideal $\mathfrak{a}^{[1/p^e]}$ of $\mathcal{O}_Y$ with

$$\mathrm{Tr}^e(F^e_*(\mathfrak{a} \cdot \omega_{Y/k})) = \mathfrak{a}^{[1/p^e]} \cdot \omega_{Y/k}.$$

Definition. (i) The test ideal of exponent $\lambda \in \mathbb{R}_{\geq 0}$ of $\mathfrak{a}$ is

$$\tau(\mathfrak{a}^\lambda) = \bigcup_{e \in \mathbb{N}_{>0}} (\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq \mathcal{O}_Y.$$

(ii) A graded sequence of ideals $\mathfrak{a}_\bullet$ in $\mathcal{O}_X$ is a family $(\mathfrak{a}_m)_{m \in \mathbb{Z}_{>0}}$ of ideals in $\mathcal{O}_X$ such that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$ for all $m, n \in \mathbb{Z}_{>0}$ and $\mathfrak{a}_m \neq (0)$ for some $m > 0$.
(iii) The asymptotic test ideal of exponent $\lambda \in \mathbb{R}_{\geq 0}$ of a graded sequence of ideals $\mathfrak{a}_\bullet$ is

$$\tau(\mathfrak{a}_\bullet^\lambda) := \bigcup_{m \in \mathbb{Z}} \tau(\mathfrak{a}_m^{\lambda/m}) \subseteq \mathcal{O}_Y.$$

Properties. (i) Have $\mathfrak{a} \subseteq \tau(\mathfrak{a})$ and $\tau(\mathfrak{a}_m) \subseteq \tau(\mathfrak{a}_\bullet^m)$ for all $m \in \mathbb{N}$.
(ii) Subadditivity Property: For all $m \in \mathbb{N}$ have

$$\tau(\mathfrak{a}_\bullet^{m\lambda}) \subseteq \tau(\mathfrak{a}_\bullet^\lambda)^m.$$

(iii) Uniform generation property.

Definition Let D be a divisor on the smooth variety Y with $H^0(Y, \mathcal{O}_Y(mD)) \neq 0$ for some $m > 0$. Define the asymptotic test ideal of exponent $\lambda \in \mathbb{R}_{\geq 0}$ associated with Y and D as

$$\tau(\lambda \cdot \|D\|) := \tau(\mathfrak{a}_\bullet^\lambda)$$

where $\mathfrak{a}_\bullet$ denotes the graded sequence of base ideals for D, i.e. $\mathfrak{a}_m$ is the image of the natural map

$$H^0(Y, \mathcal{O}(mD)) \otimes_k \mathcal{O}_Y(-mD) \to \mathcal{O}_Y.$$

Theorem (Mustaţă's uniform generation property). Let R be a k-algebra of finite type, Y an integral scheme of dimension n projective over the spectrum of R and smooth over k. Let D, E, and H be divisors on Y and $\lambda \in \mathbb{Q}_{\geq 0}$ such that

(i) $\mathcal{O}_Y(H)$ is an ample, globally generated line bundle,

(ii) $H^0(Y, \mathcal{O}_Y(mD)) \neq 0$ for some $m > 0$, and

(iii) the $\mathbb{Q}$-divisor $E - \lambda D$ is nef.

Then the sheaf $\mathcal{O}_Y(K_{Y/k} + E + dH) \otimes_{\mathcal{O}_Y} \tau(\lambda \cdot \|D\|)$ is globally generated for all $d \geq n + 1$.

- Mustaţă's original theorem requires Y projective over ground field k,
- we follow Mustaţă's proof with two modifications,
- Castelnuovo-Mumford regularity holds for the projective scheme X over R (Brodmann, Sharp),
- replace use of Fujita's vanishing theorem by application of Keeler's vanishing theorem.

Resolution of singularieties.

Resolution of singularities holds over field k **in dimension m** if for every quasi-projective variety Y over k of dimension m there exists regular variety $\tilde{Y}$ over k and projective morphism $\tilde{Y} \to Y$ which is isomorphism over the regular locus of Y.

Embedded resolution of singularities over a field k **in dimension m** holds if for every quasi-projective regular variety Y over k of dimension m and every proper closed subset Z of Y, there is a projective morphism $\pi : Y' \to Y$ of quasi-projective regular varieties over k such that $\pi^{-1}(Z)$ is the support of a normal crossing divisor and π is isomorphism over $Y \setminus Z$.

Theorem (Cossart-Piltant). Resolution of singularities and embedded resolution of singularities hold in dimension three over any perfect field.

Continuity of the envelope (ideas of the proof)

- $\{f\colon X^{\mathrm{an}} \to \mathbb{R} \mid f = -\log \|1\| \text{ for some model metric on } \mathcal{O}_X\}$ is the $\mathbb{Q}$-vector space $\mathscr{D}(X)$ of model functions,
- $N^1(\mathscr{X}/S)$ denotes space of $\mathbb{Q}$-line bundles on $\mathscr{X}$ modulo numerical equivalence in special fiber: $[\mathscr{L}] = 0 \in N^1(\mathscr{X}/S) \Leftrightarrow \deg(\mathscr{L}|_C) = 0$ for all curves C in $\mathscr{X}_s$.
- If $\|\ \|$ is model metric on L then

$$c_1(L, \|\ \|) \in Z^{1,1}(X) := \varprojlim_{\mathscr{X} \text{ model}} N^1(\mathscr{X}/S) \otimes_{\mathbb{Q}} \mathbb{R}.$$

- Call $\theta \in Z^{1,1}(X)$ semipositive if θ is induced by a nef element in $N^1(\mathscr{X}/S)$ for some model $\mathscr{X}$.
- Have $dd^c\colon \mathscr{D}(X) \to Z^{1,1}(X),\ f \mapsto c_1(\mathcal{O}_X, \|\ \|_{\mathrm{triv}} \cdot e^{-f})$.

- The space of θ-psh model functions is defined as

$$\mathrm{PSH}_{\mathscr{D}}(X,\theta) = \{f \in \mathscr{D}(X) \mid \theta + dd^c f \text{ is semipositive}\}.$$

- Say that $\theta \in Z^{1,1}(X)$ has ample de Rham class if restriction to X of representative of θ is $\mathbb{R}_{>0}$-linear combination of classes induced by ample line bundles on X.
- Fix $\theta \in Z^{1,1}(X)$ with ample de Rham class. Given $u\colon A^{\mathrm{an}} \to \mathbb{R}$ continuous define θ-psh envelope $P_\theta(u)\colon X^{\mathrm{an}} \longrightarrow \mathbb{R}$ by

$$P_\theta(u) = \sup_{\text{pointwise}} \{\varphi \mid \varphi \in \mathrm{PSH}_{\mathscr{D}}(X,\theta) \wedge \varphi \le u\}.$$

- Continuity of θ-psh enevlope is equivalent continuity of semipositive envelope.
- We have $P_\theta(u) - v = P_{\theta + dd^c v}(u - v)$ for each $v \in \mathscr{D}(X)$.
- By last equality it suffices to show continuity of $P_\theta(0)$.

- **Proposition (Boucksom, Favre, Jonsson)** Let L be ample line bundle on X, $\mathscr{L}$ extension to a model $\mathscr{X}$ of X and $\theta = c_1(\mathscr{L}, \| \ \|_{\mathscr{L}}) \in Z^{1,1}(X)$. For $m > 0$ let

$$\mathfrak{a}_m := \mathrm{Im}\left(H^0(\mathscr{X}, \mathscr{L}^{\otimes m}) \otimes_{K^\circ} \mathscr{L}^{\otimes -m} \longrightarrow \mathcal{O}_{\mathscr{X}}\right)$$

be the m-th base ideal of $\mathscr{L}$. Let $\varphi_m := m^{-1} \log |\mathfrak{a}_m|$ be $1/m$-times model function determined by $\mathscr{O}(E)$ for exceptional divisor E of blowup of $\mathscr{X}$ along vertical ideal $\mathfrak{a}_m$. Then $\varphi_m \in \mathrm{PSH}_{\mathscr{D}}(X, \theta)$ and pointwise on X^{an}

$$\lim_m \varphi_m = \sup_m \varphi_m = P_\theta(0).$$

- Embedded resolution of singularities shows that projective models are dominated by SNC-models.
- Assumption that X is of geometric origin construction says that X comes by base change from variety over perfect field k with a fibration to curve.

Proposition (GJKM). There exist normal curve B over a perfect field k, closed point $b \in B^{(1)}$, a projective regular integral scheme $\mathscr{X}_B$ over B, and line bundles $\mathscr{L}_B$ and $\mathscr{A}_B$ over $\mathscr{X}_B$ such that there exist

- a flat morphism $h\colon \operatorname{Spec} K^\circ \to \operatorname{Spec} \mathcal{O}_{B,b} \to B$,
- an isomorphism $\mathscr{X}_B \otimes_B \operatorname{Spec} K^\circ \xrightarrow{\sim} \mathscr{X}$,
- an isomorphism $h^*\mathscr{L}_B \xrightarrow{\sim} \mathscr{L}$ over the isomorphism above,
- and an isomorphism $\mathscr{A}_B|_{\mathscr{X}_{B,\eta}} \xrightarrow{\sim} \mathscr{L}_B|_{\mathscr{X}_{B,\eta}}$ where η is the generic point of B and the line bundle $\mathscr{A}_B$ on $\mathscr{X}_B$ is ample.

Read all isomorphisms above as identifications. Have cartesian diagram

$$\begin{array}{ccccc} \mathscr{X} & \xrightarrow{g} & \mathscr{X}_B & & \\ \downarrow & & \downarrow & \searrow & \\ \operatorname{Spec} K^\circ & \xrightarrow{h} & B & \longrightarrow & \operatorname{Spec} k. \end{array}$$

- Observe $\mathscr{X}_B$ is smooth variety over k. Write

$$\mathfrak{a}_{B,m} = \mathrm{Im}\big(H^0(\mathscr{X}_B, \mathcal{L}_B^{\otimes m}) \otimes_k \mathscr{L}_B^{\otimes -m} \to \mathcal{O}_{\mathscr{X}_B}\big)$$

  for m-th base ideal of $\mathscr{L}_B$ with $\mathfrak{a}_m = g^*\mathfrak{a}_{B,m}$ for all $m \in \mathbb{Z}_{>0}$.
- $\mathfrak{a}_{B,\bullet} = (\mathfrak{a}_{B,m})_{m>0}$ defines graded sequence of ideals. Let $\mathfrak{b}_{B,m} := \tau(\mathfrak{a}_{B,\bullet}^m)$ be asymptotic test ideal of exponent m and define $\mathfrak{b}_m := g^*\mathfrak{b}_{B,m}$.
- These ideals have the following properties:
  - (i) We have $\mathfrak{a}_m \subset \mathfrak{b}_m$ for all $m \in \mathbb{Z}_{>0}$.
  - (ii) We have $\mathfrak{b}_{ml} \subset \mathfrak{b}_m^l$ for all $l, m \in \mathbb{Z}_{>0}$.
  - (iii) There is $m_0 \geq 0$ such that $\mathscr{A}^{\otimes m_0} \otimes \mathscr{L}^{\otimes m} \otimes \mathfrak{b}_m$ is globally generated for all $m > 0$.
- Following BFJ, it is now a formal consequence of these properties that $\lim_m \varphi_m = P_\theta(0)$ converges uniformly.

Thank you for your attention!