Heights of cycles in toric varieties

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Let $\alpha \in \overline{\mathbb{Q}}^{\times}$.

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$$h_{\mathrm{Weil}}(\alpha) := \sum_{v \in \mathfrak{M}_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^{+} |\alpha|_v$$

with:

- K is any number field containing α
- $\mathfrak{M}_K$ the set of places of K.

For $f \in \mathbb{C}[T, T^{-1}]$, its (*logarithmic*) *Mahler measure* is:

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One has, for f_α the minimal polynomial of α over $\mathbb{Z}$:

$$h_{\mathrm{Weil}}(\alpha) = \frac{\mathrm{m}(f_\alpha)}{\deg(f_\alpha)}.$$

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GOAL:

show a similar relation for more general height functions

Arakelov geometry of toric varieties

Adelic Arakelov geometry

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Algebraic datum:

- an *adelic field* satisfying the product formula (e.g. a global field)

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Analytic data:

- a semipositive continuous metric on the analytifications of $\mathscr{O}(D)$

Height of cycles

For a cycle Z in X:

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Examples: affine spaces, projective spaces,...

Why toric varieties?

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Toric varieties have provided a remarkably fertile testing ground for general theories [...] (Their properties make) everything much more computable and concrete than usual.

(William Fulton, *Introduction to Toric Varieties*)

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If v is archimedean,

$$(z_1, \ldots, z_n) \mapsto (-\log |z_1|, \ldots, -\log |z_n|).$$

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Proposition (Oda)

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Theorem (Burgos Gil, Philippon, Sombra)

$\overline{D}$ an "adelic" semipositive toric metrized divisor,

$$h_{\overline{D}}(X) = (n+1)! \sum_{v \in \mathfrak{M}_K} n_v \int_{\Delta_D} \vartheta_v \, d\,\mathrm{vol}\,.$$

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D hyperplane at infinity

canonical metric at each place

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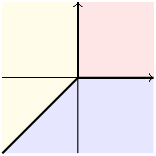

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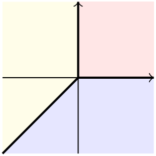

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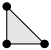

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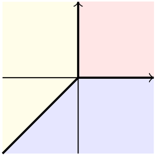

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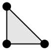

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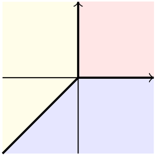

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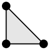

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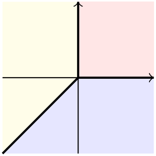

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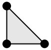

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$$\deg_D(\mathbb{P}^n) = 1 \qquad h_{\mathrm{Weil}}(\mathbb{P}^n) = 0.$$

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$$\deg_D(Y_f) = \mathrm{MV}(\Delta_D, \ldots, \Delta_D, \mathrm{NP}(f)).$$

with:

- MV a polarization of the volume of convex bodies
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what about $h_{\overline{D}}(Y_f)$?

Some notions in convex geometry

Sup-convolution

For two concave functions on polytopes

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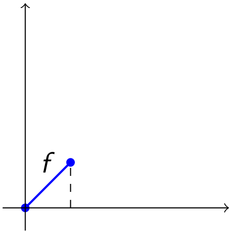
$$(f \boxplus g)(x) := \sup_{y+z=x} (f(y) + g(z)).$$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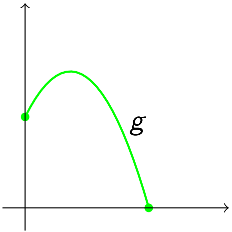
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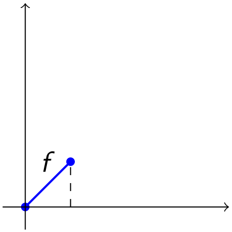
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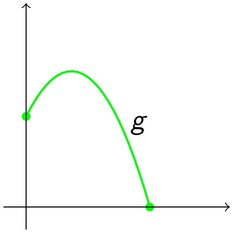
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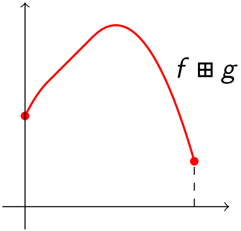
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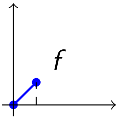
Their *mixed integral* is

$$\mathrm{MI}_\mu(g_0, \ldots, g_n) := \sum_{k=0}^{n} (-1)^{n-k} \sum_{0 \le i_0 < \cdots < i_k \le n} \int_{Q_{i_0} + \cdots + Q_{i_k}} g_{i_0} \boxplus \cdots \boxplus g_{i_k} \, d\mu.$$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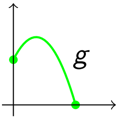
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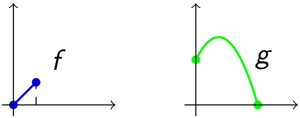
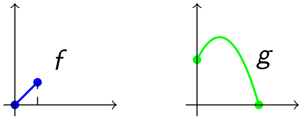


$$\mathrm{MI}_\mu(f,g) = \int_{P+Q} (f \boxplus g) d\mu - \int_P f d\mu - \int_Q g d\mu.$$



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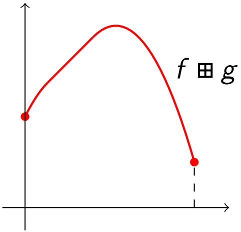

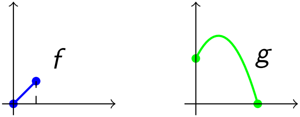

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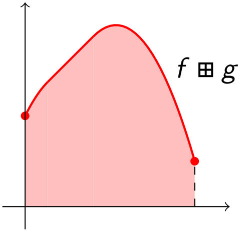

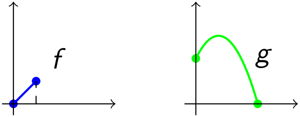

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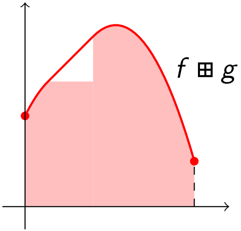

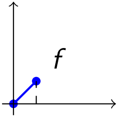

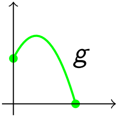

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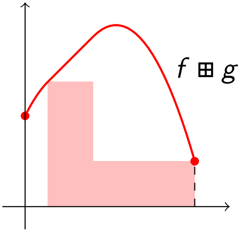

Height of hypersurfaces in toric varieties

The setting

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$$\rightsquigarrow$$

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$$\begin{aligned}\rho_{f,v} : N_{\mathbb{R}} &\to \mathbb{R},\\ u &\mapsto \int_{\mathrm{trop}_v^{-1}(u)} -\log \|f\|_x \; d\,\mathrm{Haar}_{\mathrm{Sh}(\mathrm{trop}_v^{-1}(u))}\,.\end{aligned}$$

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$$\rho_{f,v}(u) := \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} -\log\left|f\left(e^{-u_1+i\theta_1},\dots,e^{-u_n+i\theta_n}\right)\right| d\theta_1\dots d\theta_n,$$

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$$\rho_{f,v}(u) = \min_m(\langle m,u\rangle - \log|c_m|) = f^{\mathrm{trop}}(u).$$

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$$\rho_{f,v}^{\vee}(x) := \inf_{u \in N_{\mathbb{R}}} (\langle x, u \rangle - f(u))$$

is defined over $\mathrm{NP}(f)$.

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$$h_{\overline{D}}(Y) = \sum_{v \in \mathfrak{M}} n_v \, \mathrm{MI}_M(\vartheta_v, \ldots, \vartheta_v, \rho^{\vee}_{f,v}).$$

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Higher codimensions

The setting

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Suppose it has codimension k.

Proposition

D a globally generated toric divisor,

$$\deg_D(Z(f_1, \ldots, f_k)) = \mathrm{MV}(\Delta_D, \ldots, \Delta_D, \mathrm{NP}(f_1), \ldots, \mathrm{NP}(f_k)).$$

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For every place v one has $\rho_{f,v} = \rho_{g,v} = \rho_{g',v}$ but

$$h_{\mathrm{Weil}}(Z(f,g)) = 0 \qquad h_{\mathrm{Weil}} Z((f,g')) = \frac{1}{2} \log 2.$$

Upper bounds

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$$u \mapsto -\max_{\mathrm{trop}_v^{-1}(u)} \log \|f\|_x.$$

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$\overline{D}$ an "adelic" semipositive toric metrized divisor, then

$$h_{\overline{D}}(Z(f_1,\dots,f_k)) \leq \sum_{v\in\mathfrak{M}} n_v \,\mathrm{MI}_M(\vartheta_v,\dots,\vartheta_v,\mu^{\vee}_{f_1,v},\dots,\mu^{\vee}_{f_k,v}).$$