Rational vs transcendental points on analytic Riemann Surfaces

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– M will be a non compact Riemann Surface and U be a relatively compact open set on it.

– $f : M \to X(\mathbf{C})$ will be a holomorphic map with Zariski dense image.

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For every positive real number T, we are interested in understanding the following set:

$$S_{U,f}(T) := \{z \in U \ / \ f(z) \in X(\mathbf{Q}) \text{ and } h_{\mathscr{L}}(f(z)) \leq T\}$$

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In particular, we want to estimate, in terms of T, its cardinality:

$$A_{U,f}(T) := \mathrm{Card}\, S_{U,f}(T).$$

when $T \to +\infty$.

A cornerstone Theorem in this direction is the Bombieri – Pila Theorem:

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Theorem (Bombieri –Pila)

For every $\epsilon > 0$, we have that

$$A_{U,f}(T) \ll_{\epsilon} \exp(\epsilon \cdot T)$$

where the involved constants depend on ϵ, f, $\mathscr{L}$ etc. but they are independent on T.

Polynomial bounds

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Question

Can we give conditions (on M, on U, on f etc) in order to obtain that

$$A_{U,f}(T) \ll P(T) \;\; \forall T \in \mathbf{R}_{\geq 0}$$

Where $P(T) \in \mathbf{R}[T]$ is a fixed polynomial in T?
In this case we say that "we have polynomial bound".

Polynomial bounds

The literature on the problem is quickly becoming huge. We quote just some of the results:

– Masser found some results related to $\zeta(z)$;

– Boxall and Jones found some conditions on the growth of entire functions which imply the polynomial bound when f is the graph of a transcendental entire function;

– Comte and Yomdim found some conditions on the Taylor expansion of a holomorphic function which imply the polynomial bound for its graph;

– Binyamini proved the polynomial bound for the graph of a transcendental function verifying a differential equation;

– Schmidt proved the polynomial bound for the graph of elliptic functions;

– Villemont (student of Comte - Thesis which will be defended in november 2018) proved the polynomial bound for graphs of fuchsian functions.

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– For every $\alpha \in \overline{\mathbf{Q}}$ and for every $n \geq 0$ we have that $h^{(n)}(\alpha) \in \mathbf{Q}(\alpha)$;

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– if $f : \mathbf{C} \to \mathbf{C}^2$ is the graph of h and $U = \{|z| < 1\}$ then, there exists a sequence of positive integers $(N_k)_{k \in \mathbf{N}}$ such that:

$$A_{U,f}(N_k) \geq \frac{\exp(2\varphi(N_k))}{2}.$$

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The proof is similar to the construction, by Stäckel (1895), of entire functions whose value at algebraic points is algebraic.

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Theorem

Let $f : M \to X(\mathbf{C})$ and U above. Let $A > 1$ (very big), $\epsilon > 0$ (very small) and $\gamma > \frac{N}{N-1}$. Then, there exists a, unbounded, increasing sequence of real numbers r_n such that

$$\forall T \in [r_n; Ar_n] \text{ we have that } A_{U,f}(T) \leq \epsilon T^{\gamma}.$$

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It is possible to find a similar theorem for $\gamma = \frac{N}{N-1}$.

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Choose an integer $d_1 \sim (\epsilon T_1)^{\gamma/N}$.

By some forms of Siegel Lemma, we can find a non zero section $s \in H^0(\mathscr{X}, \mathscr{L}^{d_1})$ such that:
– $f^*(s)|_{S_{U,f}(T_1)} = 0$;
– $\log \|s\| \leq c \cdot T_1^{1+\gamma/N}$.

– By induction we may suppose that $f^*(s)$ vanishes on $S_{U,f}(T_n)$.

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Since $f^*(s)(w) \neq 0$, *Liouville Inequality* gives

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$$\log \|f^*(s)\|(w) \geq -C_4 \cdot T_1^{\gamma/N} \cdot T_{n+1}.$$

Since $T_{n+1} \leq A \cdot T_n$, we get:

$$C_5 \cdot T_n^{1+\gamma/N} \geq C_6 T_n^{\gamma}.$$

And this is a contradiction as soon as T_n is big enough. Thus $f^*(s)$ vanishes on $S_{U,f}(T_n)$ for every n and this is impossible.

Polynomial bounds

Consequently: for arbitrary function $f : M \to X(\mathbf{C})$, even if, in general, we have not a polynomial bound, we have a polynomial bound on infinitely many arbitrarily big intervals.

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– A "small section of a line bundle vanishing in many points"

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• To bound $A_{u,f}(T)$ we need:

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– Some point where this small section *do not vanish but it is not too small there*.

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• To bound $A_{u,f}(T)$ we need:

– A "small section of a line bundle vanishing in many points"

– Some point where this small section *do not vanish but it is not too small there*.

• Moreover, what prevents to improve the theorem above is the fact that the point where we applied the Liouville inequality, depends on T.

For this reason we introduce the following definition:

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Definition

Let $W \subseteq U$ be a subset. We will say that W is of type S with respect to f, if there exist positive constants A and a such that, for every positive integer d ad every global section $s \in H^0(\mathscr{X}, cL^d) \setminus \{0\}$ we have

$$\sup_{z \in W} \{\log \|f^*(s)\|(z)\} \geq -A \left(\log^+ \|s\| + d\right)^a$$

Where $\log^+ \|s\| := \sup_{p \in X(\mathbf{C})} \{0, \log \|s\|(p)\}$.

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Remark: One can show that, if W exists, then one must have $a \geq N + 1$

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An interesting application of this theorem is the following:

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An interesting application of this theorem is the following:

Theorem

Suppose that $f : M \hookrightarrow X(\mathbf{C})$ is the leaf of an algebraic foliation by curves (defined over $\mathbf{Q}$), and $p_0 \in M$ is a rational point which is smooth for the foliation. Then there exists a open set $p_0 \in V \subset M$ which is of type S with respect to f.

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This Corollary generalizes previous results by Binyamini and Comte –Yomdim .

Points of Type S on varieties.

Actually, it is not easy to prove the existence of a subset W of type S.

In theory, the biggest the subset is and the easier should be to find it. But, in principle (in order to obtain the consequences on the estimates on rational points), it suffices that W is just a single point.

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Let $p \in X(\mathbf{C})$. We will say that p is of type S on X if there exist positive constants A and a (depending on p) such that, for every positive integer d and every non vanishing section $s \in H^0(\mathscr{X}, \mathscr{L}^d)$ we have that

$$\log \|s\|(p) \geq -A\left(\log^+ \|s\| + d\right)^a .$$

(where $\log^+(a) := \sup\{\log(a), 0\}$). We will denote by $S(X)$ the subset of points of type S.

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(where $\log^+(a) := \sup\{\log(a), 0\}$). We will denote by $S(X)$ the subset of points of type S.

Of course a point of type S is transcendental and one can prove that the set $S(X)$ is independent on the set of the model $\mathscr{X}$ and the polarization

As a corollary of what we said we find that:

Corollary

If $f : M \to X(\mathbf{C})$ is as above and $f^{-1}(S(X)) \neq \emptyset$ then $A_{U,f}(T)$ is polynomially bound in T

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Thus we can say that "essentially all the points of $X(C)$ are of type S! (even, due to their genericity, we cannot give a single example of point of type S!)

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Another Theorem we can prove is the following:

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Let $f : M \to X(\mathbf{C})$ as above. Then, there exists a subset $W \subset M$ of type S with respect to f *if and only if* $f^{-1}(S(X))$ is full in M.

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Thus we obtain the following diagram:

$$
\begin{array}{ccc}
 & & f^{-1}(S(X)) \text{ is full in } M \\
 & & \uparrow \\
M \text{ is a leaf of a foliation} & \longrightarrow & f^{-1}(S(X)) \neq \emptyset \\
 & & \downarrow \\
 & & A_{U,f}(T) \ll Poly(T)
\end{array}
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3) For a given f as in (2), can we estimate how big is the set of T's for which $A_{U,f}(T)$ is "big"?

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3) For a given f as in (2), can we estimate how big is the set of T's for which $A_{U,f}(T)$ is "big"?

4) (In the spirit of Lang conjecture) Since we expect that, for varieties of general type, the set of rational points is "small", is it possible that, in this case, the set $S(X)$ is, instead, big? For instance the set $X(\mathbf{C}) \setminus S(X)$ has Hausdorff dimension $2N - 2$ in this case? (its Hausdorff dimension cannot be smaller because it contains all the varieties defined over $\overline{\mathbf{Q}}$.

tak for din opmærksomhed !