Mod-affine varieties in Arakelov geometry

Intercity Seminar on Arakelov Geometry

University of Copenhagen

September 3, 2018

Jean-Benoît Bost

(Université Paris-Sud, Orsay)

joint work with François Charles (Orsay)

Motivation / Question :

In "classical" algebraic geometry, affine schemes play a basic role

What is their natural counterpart in Arakelov geometry?

Is it possible to develop Arakelov geometry in a way similar to algebraic geometry (à la Weil, Serre, Chevalley, Grothendieck, ...) by defining suitable "affine objects" and then glueing them?

This turns out to be almost possible by combining two ingredients

(i) a good understanding of "hermitian quasi-coherent sheaves" over arithmetic curves $\operatorname{Spec} \mathcal{O}_K$:

infinite dimensional geometry of numbers

θ- invariants of euclidean lattices

$$\left(h^0_\theta(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2_{\overline{E}}} \right)$$

(ii) some (not so well-) known results of algebraic geometry (EGA style) and of analytic geometry (Grauert style), to handle "relative Arakelov varieties" $\widetilde{\mathcal{X}} \longrightarrow \overline{\operatorname{Spec} \mathcal{O}_K}$.

Actually : "mod-affine" instead of "affine"

Technical gain : allows one to work without any regularity or reducedness assumption.

- Geometric counterpart

k base field

C smooth, projective, geometrically connected

$K := k(C)$ function field

Theorem ($\Leftarrow$ Serre affineness criterion + ...)

For any $\pi : X \longrightarrow C$ scheme, separated of finite type over C,

TFCAE:

(i) the scheme X is affine;

(ii) the morphism π is affine and for every coherent Ideal $\mathcal{I}$ in $\mathcal{O}_X$,

$$H^1(C, \pi_* \mathcal{I}) = 0 ;$$

(iii) the morphism π is affine and for every coherent $\mathcal{O}_X$-Module $\mathcal{F}$ over X,

$$H^1(C, \pi_* \mathcal{F}) = 0 .$$

If we know (a) what a relatively affine $\pi : X \to C$ is, and (b) how to define $\pi_* \mathcal{F}$ for $\mathcal{F}$ coherent over X and the vanishing of $H^1(C, \pi_* \mathcal{F})$, then we may recover the affine k-schemes.

I Mod-affine schemes

Mod-affine S-schemes

$S :=$ Noetherian base scheme

Theorem (Goodman - Hartshorne 1969, Lütkebohmert 1990 + ε):

For any scheme, separated and of finite type over S, $\pi : X \longrightarrow S$, TFCAE:

MA_1 : For every coherent $\mathcal{O}_X$ - Module $\mathcal{F}$ over X and every $i > 0$, $R^i \pi_* \mathcal{F}$ is coherent;

MA_2 : For every coherent $\mathcal{O}_X$ - Module $\mathcal{F}$ over X, $R^1 \pi_* \mathcal{F}$ is coherent;

MA_3 : There exist :
- a scheme affine of finite type over S $\pi_A : A \longrightarrow S$
- $F \hookrightarrow A$ finite over S
- $\nu : X \longrightarrow A$ a proper S-morphism

such that $X \setminus \nu^{-1}(F) \xrightarrow[\sim]{\nu} A \setminus F$

MA_4 : Same as MA_3, and $\mathcal{O}_A \xrightarrow{\sim} \nu_* \mathcal{O}_X$

In MA_4, $A \xrightarrow{\sim} \mathrm{Spec}_S\, \pi_* \mathcal{O}_X$ is uniquely determined and ν is a so-called modification of the affine S-scheme A.

When MA_{1-4} hold, π is called mod-affine, and X is called mod-affine over S.

Properties of mod-affine S-schemes

Proposition. Consider S' a Noetherian S-scheme

$\pi : X \to S$ separated of finite type

$\pi' : X_{S'} := X \times_S S' \to S'$

Then:

$$\pi \text{ mod-affine} \Rightarrow \pi' \text{ mod-affine}$$

Conversely, if S' is faithfully flat over S,

$$\pi' \text{ mod-affine} \Rightarrow \pi \text{ mod-affine}.$$

Theorem. Consider a proper morphism φ of S-schemes, separated and of finite type over S

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S \arrow[r, equal] & S \end{tikzcd}.$$

1) If there exists $P \hookrightarrow X$, proper over S, such that $\varphi : X' \setminus \varphi^{-1}(P) \to X \setminus P$ is finite, then

$$\pi \text{ mod-affine} \Rightarrow \pi' \text{ mod-affine}.$$

2) If φ is surjective, then

$$\pi' \text{ mod-affine} \Rightarrow \pi \text{ mod-affine}$$

Corollary: If φ is finite and surjective

$$\pi \text{ mod-affine} \Leftrightarrow \pi' \text{ mod-affine}$$

Constructions of mod-affine S-schemes

Theorem (Grauert ampleness criterion): Consider a proper S-scheme $\pi : X \to S$ and a line bundle L over X.

TFCAE:

(i) L is π-ample;

(ii) $\mathbb{V}(L) := \mathrm{Spec}_X \bigoplus_{m \in \mathbb{N}} L^{\otimes m}$ is mod-affine over S.

Theorem (Goodman, Hartshorne). Let X be a scheme proper over some field k, and $Y \hookrightarrow X$ an effective Cartier divisor.

If $N_Y X$ $(:= \mathcal{O}_X(Y)_{|Y})$ is ample, then $\mathcal{U} := X \setminus Y$ is mod-affine (over k).

Mod-affine C-schemes and k-schemes

Consider k, C, and $\pi : X \longrightarrow C$ (separated of finite type) as above

Proposition. TFCAE

(i) X is mod-affine over k;

(ii) π is mod-affine and, for every coherent $\mathcal{O}_X$-Module $\mathcal{F}$ over X

$$\dim_k H^1(C, \pi_* \mathcal{F}) < +\infty .$$

Proposition. Let $\pi_1 : X_1 \longrightarrow C$ and $\pi_2 : X_2 \longrightarrow C$ be affine (of finite type)

Then:

X_1 and X_2 mod-affine over k $\Longrightarrow$ $X := X_1 \times_C X_2$ mod-affine over k

II Mod – Stein analytic spaces and compact subsets

Mod-Stein analytic spaces

Theorem (Grauert; R. Narasimhan)

For any $\mathbb{C}$-analytic space X (with Hausdorff, second countable, topology), TFCAE

MS_1: For every coherent $\mathcal{O}_X^{an}$-Module $\mathcal{F}$ over X, and every $i>0$,

$$\dim_{\mathbb{C}} H^i(X,\mathcal{F}) < +\infty \, ;$$

MS_2: For every coherent Ideal $\mathcal{I}$ in $\mathcal{O}_X^{an}$,

$$\dim_{\mathbb{C}} H^1(X,\mathcal{I}) < +\infty \, ;$$

MS_3: There exists a Stein analytic space A, a proper analytic map $\nu : X \longrightarrow A$, and a finite subset F of A such that

$$\nu : X \setminus \nu^{-1}(F) \xrightarrow{\sim} A \setminus F.$$

MS_4: Same as MS_3 and $\mathcal{O}_A^{an} \xrightarrow{\sim} \nu_* \mathcal{O}_X^{an}$

MS_5: There exists a proper continuous function $\varphi : X_{red} \longrightarrow \mathbb{R}_+$ which is strongly p.s.h. outside some compact subset.

In MS_4, A is uniquely determined and ν is a so-called modification of the Stein space A.

When MS_{1-5} hold, X is called a mod-Stein analytic space.

Mod-Stein compact subsets of an analytic space

Theorem Let X be a $\mathbb{C}$-analytic space and let K be a compact subset of X_{red}. TFCAE:

CMS_1: There exists a basis $(U_i)_{i \in \mathbb{N}}$ of mod-Stein open neighborhoods of K in X_{red}

CMS_2: There exists:
- an analytic space A
- a proper analytic map $\nu : X \longrightarrow A$
- $\underset{\text{finite}}{F} \hookrightarrow \underset{\text{compact Stein}}{L} \hookrightarrow A$

such that:

$$L = \nu^{-1}(K) \quad \text{and} \quad \nu : X \setminus \nu^{-1}(F) \xrightarrow{\sim} A \setminus L.$$

CMS_3: There exists an open neighborhood $\mathcal{U}$ of K in X_{red} and some continuous (psh) function $\varphi : \mathcal{U} \longrightarrow \mathbb{R}_+$ such that:

$$K = \varphi^{-1}(0) \quad \text{and} \quad \varphi \text{ strictly psh on } \mathcal{U} \setminus K (:= \varphi^{-1}(\mathbb{R}_+^\times))$$

When CMS_{1-3} hold, K is a called a mod-Stein compact subset of X.

N.B: $K = \bigcap_{n \geq 0} \mathcal{U} \subset \dots \subset \underset{\text{mod-Stein open}}{U_n} \subset U_{n-1} \subset \dots \subset U_0$ $\qquad U_i := \varphi^{-1}(2^{-i}\varepsilon)$

for any coherent sheaf $\mathcal{F}$ on some open neighbd of K in X, and any $i > 0$:

$H^i(K, \mathcal{F}) \simeq \varinjlim_n H^i(U_n, \mathcal{F}) \xrightarrow{\sim} H^i(U_n, \mathcal{F})$ for $n \gg 0$; finite dimensi

Examples (Grauert):

1) $\mathcal{X}$ $\mathbb{R}$-scheme (separated of finite type)

$X := \mathcal{X}_{\mathbb{C}}^{an}$

any compact subset K of $\mathcal{X}(\mathbb{R})$ is a Stein compact subset of $\mathcal{X}_{\mathbb{C}}^{an}$

2) $\left\{ \begin{array}{ll} X & \text{compact analytic space} \\ L & \text{line bundle on } X \\ \| \ \| & \text{continuous metric on } L_{|X_{red}} \end{array} \right.$

$\leadsto$ $K := D_{\bar{L}}$ = unit disk bundle $\hookrightarrow \mathbb{V}(L)_{red}$ (total space of $L^{\vee}_{|X_{red}}$)

K is mod-Stein $\iff$ L is ample and $c_1(L_{|X_{red}}, \| \ \|) \geq 0$.

IV Infinite dimensional geometry of numbers: ϑ- invariants and nuclear spaces

Recall : <u>Geometry of numbers</u> , <u>Arakelov style</u>

K number field, $\mathcal{O}_K$, $\pi : \operatorname{Spec} \mathcal{O}_K \longrightarrow \operatorname{Spec} \mathbb{Z}$

Hermitian vector bundle over $\operatorname{Spec} \mathcal{O}_K$

$$\overline{E} := \big(E, (\|\cdot\|_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}} \big)$$

E: $\mathcal{O}_K$ - module, projective of finite type

$\|\cdot\|_\sigma$: hermitian metric on $E_\sigma := E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}$

+ compatible with c.c.

notably, when $\mathcal{O}_K = \mathbb{Z}$, $\overline{E} = (E, \|\ \|)$ is an Euclidean lattice

$E \simeq \mathbb{Z}^r$, $\|\ \|$: euclidean norm on $E_{\mathbb{R}} \simeq \mathbb{R}^r$

$\rightsquigarrow \pi_* \overline{E} := \big(E, \|\ \|^2 := \sum_{\sigma : K \hookrightarrow \mathbb{C}} \|\cdot\|_\sigma^2 \big)$ over $\operatorname{Spec} \mathbb{Z}$

$E \simeq \mathbb{Z}^{[K:\mathbb{Q}]\, \mathrm{rk}\, E}$

Arakelov degree

over $\operatorname{Spec} \mathbb{Z}$ $\qquad \widehat{\deg}\, \overline{E} := -\log \operatorname{covol} \overline{E}$

in general

$$:= \widehat{\deg}\, \pi_* \overline{E} - \mathrm{rk}\, E \ \widehat{\deg}\, \pi_* \overline{\mathcal{O}}_K$$
$$= \widehat{\deg}\, \pi_* \overline{E} + \mathrm{rk}\, E \ \tfrac{1}{2} \log |\Delta_K|$$

ϑ – invariants

Hecke, ..., Rœssler – Morishita, ..., Banaszczyk, ...
van der Geer – Schoof, Groenewegen

over $\mathrm{Spec}\,\mathbb{Z}$, $\quad h^0_\vartheta(\overline{E}) := \log \sum_{v \in E} e^{-\pi \|v\|^2}$

$$h^1_\vartheta(\overline{E}) := h^0_\vartheta(\overline{E}^\vee)$$

in general $\quad h^i_\vartheta(\overline{E}) := h^i_\vartheta(\pi_* \overline{E})$

N.B. : tout se passe comme si $\; h^i_\vartheta(\overline{E}) := \dim_? H^i(\overline{\mathrm{Spec}\,\mathcal{O}_K}, \overline{E})$.

↳ real valued dimension

For instance :

- $h^i_\vartheta(\overline{E}) \in \mathbb{R}_+$

- Poisson – Riemann – Roch – Hecke $\quad \overline{\omega_\pi} := (\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z}), \| \mathrm{tr} \|_{\sigma} =$

$$h^0_\vartheta(\overline{E}) - h^1_\vartheta(\overline{E}) = h^0_\vartheta(\overline{E}) - h^0_\vartheta(\overline{E} \otimes \overline{\omega_\pi}) = \widehat{\deg}\,\overline{E} - \mathrm{rk}\,E\, \tfrac{1}{2} \log |\Delta$$

- sub-additivity $\quad 0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$

$$h^0_\vartheta(\overline{F}) \leq h^0_\vartheta(\overline{E}) + h^0_\vartheta(\overline{G})$$

Infinite dimensional geometry of number

see JBB arXiv: 1512:08946 [v2]

Ind - hermitian vector bundle over Spec $\mathcal{O}_K$:

$\overline{F} := (F, (\|\cdot\|_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ where F is a countably generated projective $\mathcal{O}_K$-mod

$\| \|_\sigma$ hermitian norm on F_σ + compat. with c.c

Pro - hermitian vector bundle over Spec $\mathcal{O}_K$:

$\hat{\overline{E}} := (\hat{E}, (E_\sigma^{Hilb}, \|\cdot\|_\sigma, i_\sigma)_{\sigma: K \hookrightarrow \mathbb{C}})$ where :

. $\hat{E}$ is a topological $\mathcal{O}_K$-module $\simeq \mathrm{Hom}_{\mathcal{O}_K}(F, \mathcal{O}_K)$ (pointwise cv) with F as above

. $(E_\sigma^{Hilb}, \|\cdot\|_\sigma)$ complex Hilbert space

+ compat. with c.c

. $i_\sigma : E_\sigma^{Hilb} \hookrightarrow \hat{E} \hat{\otimes}_\sigma \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}(F_\sigma, \mathbb{C})$

continuous injective with dense image.

$$E_{\mathbb{R}}^{Hilb} := \left(\bigoplus_\sigma E_\sigma^{Hilb}\right)^{c.c.} \hookrightarrow \hat{E}_{\mathbb{R}} := \hat{E} \hat{\otimes}_{\mathbb{Z}} \mathbb{R} \hookleftarrow \hat{E}$$

$$\|v\|^2 := \sum_\sigma \|v\|_\sigma^2$$

Examples and constructions

1) $R \in \mathbb{R}_+^*$

arithmetic Hardy space $\widehat{\mathcal{H}}(R) := (\mathbb{Z}[[X]], \mathcal{H}^2(\mathring{D}(0,R)), i)$

$$\mathbb{Z}[[X]] \hookrightarrow \mathbb{C}[[X]] \hookleftarrow \mathcal{H}^2(\mathring{D}(0,R)) := \Big\{ \underbrace{\sum_{n \in \mathbb{N}} a_n X^n}_{= f} \;\Big|\; \underbrace{\sum_{n \in \mathbb{N}} |a_n|^2 R^{2n}}_{= \|f\|^2} < +\infty \Big\}$$

2) $\overline{E}_\bullet := \overline{E}_0 \xleftarrow{q_1} \overline{E}_1 \xleftarrow{q_2} \overline{E}_2 \twoheadleftarrow \cdots$

projective system of surjective admissible morphisms of hermitian vector bundles

may define $\widehat{E} := \varprojlim_i \overline{E}_i$ as the pro-Hermitian vector bundle attached to

$$\widehat{E} := \varprojlim_i E_i := \{ (x_i) \in \prod_i E_i \mid \forall i > 0,\ q_i(x_i) = x_{i-1} \}$$

$$E_\sigma^{Hilb} := \varprojlim_i{}^{Hilb} (E_{i,\sigma}, \| \|_{E_{i,\sigma}}) := \{ \underbrace{(x_i)}_{x} \in \prod_i E_{i,\sigma} \mid \text{" "} \text{ and } \underbrace{\lim_{i \to +\infty} \|x_i\|_{E_{i,\sigma}}}_{= \|x\|_\sigma^{Hilb}} < +\infty \}$$

3) $\mathcal{U} \subset \widehat{E}$ open saturated submodules (over $\mathcal{O}_K$)

$\rightsquigarrow$ $E_{\mathcal{U}} := \widehat{E} / \mathcal{U}$ finitely generated projective $\mathcal{O}_K$-module

$$\begin{tikzcd} E_\sigma^{Hilb} \arrow[r, hook] \arrow[dr, "\text{quotient}"'] & \widehat{E}_\sigma := \widehat{E} \hat{\otimes}_\sigma \mathbb{C} \arrow[d] \\ \| \|_\sigma & E_{\mathcal{U},\sigma} \\ & \| \|_{\mathcal{U},\sigma} \end{tikzcd}$$

$\overline{E}_{\mathcal{U}} := (E_{\mathcal{U}}, (\| \|_{\mathcal{U},\sigma})_{\sigma : K \hookrightarrow \mathbb{C}})$

ϑ-invariants of pro-Hermitian vector bundles

$$\overline{h}^0_\vartheta(\widehat{\overline{E}}) := \liminf_{\mathcal{U}} h^0_\vartheta(\overline{E}_{\mathcal{U}}) \qquad \in [0, +\infty]$$

$$\underline{h}^0_\vartheta(\widehat{\overline{E}}) := \log \sum_{v \in E^{Hilb}_{\mathbb{R}} \cap \widehat{E}} e^{-\pi \|v\|^2} \qquad \in [0, +\infty]$$

$$\overline{\mathcal{O}}(\lambda) := (\mathcal{O}_K, \| 1 \|_\sigma := e^{-\lambda}), \qquad \lambda \in \mathbb{R}$$

Definition and theorem (JBB):

1) $\widehat{\overline{E}}$ is ϑ-finite when, for every $\lambda \in \mathbb{R}$,

$$\overline{h}^0_\vartheta(\widehat{\overline{E}} \otimes \overline{\mathcal{O}}(\lambda)) = \underline{h}^0_\vartheta(\widehat{\overline{E}} \otimes \overline{\mathcal{O}}(\lambda)) < +\infty$$
$$=: h^0_\vartheta(\widehat{\overline{E}} \otimes \overline{\mathcal{O}}(\lambda))$$

2) If $\widehat{\overline{E}} = \varprojlim_i \overline{E}_i$, with $\overline{E}_\cdot : \overline{E}_0 \overset{q_1}{\twoheadleftarrow} \overline{E}_1 \overset{q_2}{\twoheadleftarrow} \overline{E}_2 \twoheadleftarrow \cdots$ as above and if, for every $\lambda \in \mathbb{R}$,

$$\sum_{i \geq 1} h^0_\vartheta(\overline{\ker q_i} \otimes \overline{\mathcal{O}(\lambda)}) < +\infty,$$

then $\widehat{\overline{E}}$ is ϑ-finite, and for every $\lambda \in \mathbb{R}$

$$h^0_\vartheta(\widehat{\overline{E}} \otimes \overline{\mathcal{O}(\lambda)}) = \lim_{i \to +\infty} h^0_\vartheta(\overline{E}_i \otimes \overline{\mathcal{O}(\lambda)})$$

N.B.: $\widehat{\overline{E}}$ ϑ-finite $\Rightarrow$ $\gamma^0(\widehat{\overline{E}}) := \widehat{E} \cap E_{\mathbb{R}}$ countably generated projective $\mathcal{O}_K$-module

ϑ - invariants of ind - Hermitian vector bundles over Spec $\mathbb{Z}$

Basic definitions : $\overline{F} := (F ; \| \cdot \|)$ where $F \simeq \mathbb{Z}^{(I)}$, $I \subset \mathbb{N}$

$\| \ \|$ euclidean norm on $F_{\mathbb{R}}$

$$h^0_\vartheta(\overline{F}) := \log \sum_{f \in F} e^{-\pi \|f\|^2}$$

$$\begin{cases} \overline{h}^1_\vartheta(\overline{F}) := \overline{h}^0_\vartheta(\overline{F}^\vee) \\ \underline{h}^1_\vartheta(\overline{F}) := \underline{h}^0_\vartheta(\overline{F}^\vee) \end{cases}$$

pro - hermitian vector bundle over Spec $\mathbb{Z}$

Generalization I :

- F may be any countably generated $\mathbb{Z}$ - module ($F^{\vee\vee}$ is then a projective countably generated $\mathbb{Z}$-module of dual $F^\vee$!!)
- $\| \cdot \|$ may be a Euclidean semi - norm on $F_{\mathbb{R}}$ (by taking decreasing limits of Euclidean norms)

Then $\overline{F}$ is said to be ϑ^1- finite when

$$\forall \lambda \in \mathbb{R}, \quad \overline{h}^1_\vartheta(\overline{F} \otimes \overline{\mathcal{O}(\lambda)}) = \underline{h}^1_\vartheta(\overline{F} \otimes \mathcal{O}(\lambda)) < +\infty$$

Generalization II   quasi-coherent A-sheaf over Spec $\mathbb{Z}$

$$\widetilde{F} := (F, (\|\cdot\|_i)_{i\in\mathbb{N}})$$

countably generated $\mathbb{Z}$-module

Euclidean semi-norms on $F_{\mathbb{R}}$ s.t. $\|\cdot\|_{i+1} \leq c_i \|\ \|_i$ up to equivalence

equivalent data:

$$\iota : F \longrightarrow F_{\mathbb{R}}^{top} := \varinjlim_i (F_{\mathbb{R}}, \|\ \|_i)^{\text{completed}}$$

D.F-space

$\widetilde{F}$ is θ^1-finite $\overset{\text{def}}{\iff}$ $\exists\, i$, $(F, \|\ \|_i)$ θ^1-finite

$\widetilde{F}$ is conuclear $\overset{\text{def}}{\iff}$ for any i, $\exists\, i' > i$, $\operatorname{Tr} \dfrac{\|\cdot\|_{i'}^2}{\|\cdot\|_i^2} < +\infty$

$\iff$ $F_{\mathbb{R}}^{top}$ is nuclear

Theorem: Let $\widetilde{F} = (F, F_{\mathbb{R}}^{top}, \iota)$ be a conuclear quasi-coherent A-sheaf over Spec $\mathbb{Z}$. If $\widetilde{F}$ is θ^1-finite, then, for any neighborhd U of 0 in $F_{\mathbb{R}}^{top}$, there exists a finitely generated submodule Φ of F such that

$$F_{\mathbb{R}}^{top} = \underbrace{\iota(\phi)_{\mathbb{R}}}_{\substack{\mathbb{R}\text{ vector space} \\ \text{generated by } \iota(\phi)}} + \iota(F) + U$$

Key lemma

$$\begin{cases} F \simeq \mathbb{Z}^N \\ \|.\|' \leqslant \|.\| \quad \text{Euclidean norms on } F_{\mathbb{R}} \end{cases}$$

If $h^1_\theta(F, \| \|) \leqslant 1/2$, then

$$\pi \underbrace{e(F, \|.\|')}_{\text{covering radius of } (F, \|.\|')}{}^2 \leqslant 7\, h^1_\theta(F, \| \|) + \mathrm{Tr} \frac{\|.\|'^2}{\|.\|^2}.$$

$$:= \min \{ r \in \mathbb{R}_+ \mid F_{\mathbb{R}} = F + \overline{B}_{\| \|'}(0, r) \}$$

N.B.:
- does not involves N !
- compare with Milos lemma, Sazonov topology, etc...

IV Mod_affine A_pairs

Definitions and constructions

- an **A-pair** is a pair $\tilde{\mathcal{X}} := (\mathcal{X}, K)$ where $\mathcal{X}$ is a scheme, separated of finite type over Spec $\mathbb{Z}$ and K is a compact subset of $\mathcal{X}(\mathbb{C})$, invariant under complex conjugation.

- Fact: with the above notation, if $\pi := \mathcal{X} \longrightarrow \operatorname{Spec}\mathbb{Z}$, for any coherent sheaf $\mathcal{F}$ on $\mathcal{X}$,

$$\pi_{\tilde{\mathcal{X}}*}\mathcal{F} := \Big(\underbrace{\pi_*\mathcal{F}}_{\Gamma(\mathcal{X},\mathcal{F})},\ \underbrace{\Gamma(K, \mathcal{F}_{\mathbb{C}}^{an})}_{= \varinjlim_{K\subset U \hookrightarrow \mathcal{X}(\mathbb{C})} \Gamma(U, \mathcal{F}_{\mathbb{C}}^{an})}\Big)$$

is a conuclear quasi-coherent A-sheaf on Spec $\mathbb{Z}$.

- $\tilde{\mathcal{X}} = (\mathcal{X}, K)$ relatively mod-affine $\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \mathcal{X} \text{ mod-affine over } \mathbb{Z} \\ K \text{ mod-Stein compact subset in } \mathcal{X}_{\mathbb{C}}^{an} \end{cases}$

  $\tilde{\mathcal{X}}$ arithmetically mod-affine $\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \tilde{\mathcal{X}} \text{ relatively mod-affine} \\ \text{for any coherent sheaf } \mathcal{F} \text{ over } \mathcal{X},\ \pi_{\tilde{\mathcal{X}}*}\mathcal{F} \text{ is } \mathcal{D}'\text{-finite} \end{cases}$

Mod-affine A-pairs and arithmetic ampleness

$\mathcal{X}$ projective scheme over $\operatorname{Spec} \mathbb{Z}$

(i) à la Gravel:

L line bundle over $\mathcal{X}$, $\|.\|$ continuous metric on $L_{|\mathcal{X}(\mathbb{C})}$ inv. under c.c.

Def.: $\overline{L} := (L, \|.\|)$ is arithmetically ample when

$$\widetilde{\mathbb{V}}(\overline{L}) := (\mathbb{V}(L), D_{\overline{L}_{\mathbb{C}}})$$

is arithmetically mod-affine. $\leftarrow$ unit disc bundle

Theorem. When $\mathcal{X}_{\mathbb{Q}}$ is regular, $\overline{L}$ is arithmetically ample in the above sense iff it is arithmetically ample à la Zhang.

(ii) à la Goodman-Hartshorne

$\mathcal{Y} \hookrightarrow \mathcal{X}$ effective Cartier divisor

$\|.\|$ continuous metric on $\mathcal{O}(\mathcal{Y})_{|\mathcal{X}(\mathbb{C})}$, invariant under complex conjugation

$g_{\mathcal{Y}_{\mathbb{C}}} := -\log \|1_{\mathcal{O}(\mathcal{Y})|\mathcal{X}_{\mathbb{C}}}\|$

Theorem: If $(\mathcal{O}(\mathcal{Y}), \|\ \|)_{|\mathcal{Y}}$ is arithmetically ample on $\mathcal{Y}$ and $c_1(\mathcal{O}(\mathcal{Y})_{|\mathcal{X}(\mathbb{C})}, \|\ \|) \geq$
then the A-pair

$$\widetilde{\mathcal{U}} := (\mathcal{U}, K),$$

defined by $\mathcal{U} := \mathcal{X} \setminus \mathcal{Y}$ and $K := \{x \in \mathcal{X}(\mathbb{C}) \mid \|1_{\mathcal{O}(\mathcal{Y})}(x)\| \geq 1\}$,
is arithmetically mod-affine

$\Updownarrow$
$g_{\mathcal{Y}_{\mathbb{C}}} \leq 0$

Example: $(\mathbb{A}^1_{\mathbb{Z}}, K)$ arithmetically mod-affine $\Longleftrightarrow$ transfinite diameter of $K < 1$