Géométrie d'Arakelov sur une courbe adélique

(joint work with Atsushi Moriwaki)

Intercity seminar in Arakelov geometry 2018

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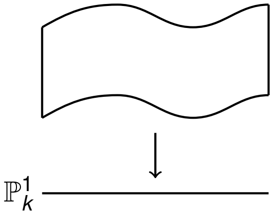
Arithmetic geometry

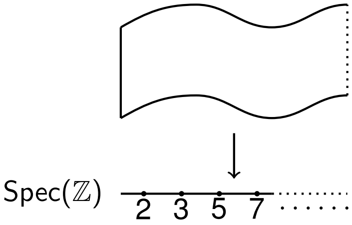
Objects

Schemes of finite type over $\mathrm{Spec}\,\mathbb{Z}$ (Diophantine systems)

Comparison with algebraic geometry



Algebraic geometry



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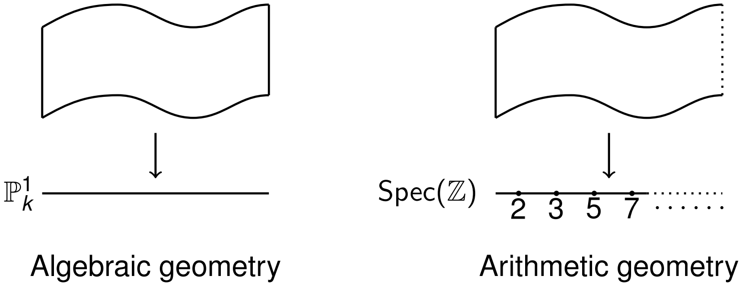
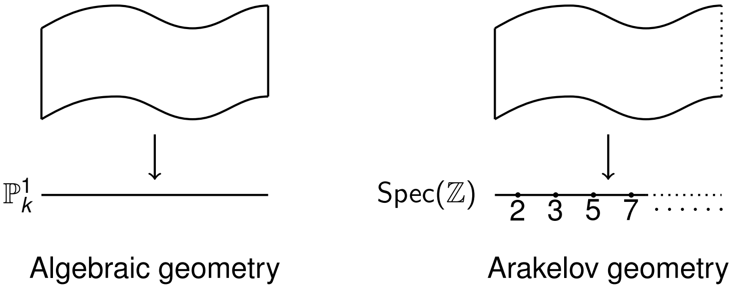


Special features of arithmetics

- No base field
- No “compact arithmetic curve”

Arakelov's philosophy

- "Compactify" arithmetic varieties by analytic objects
- Combine analysis and algebraic geometry methods



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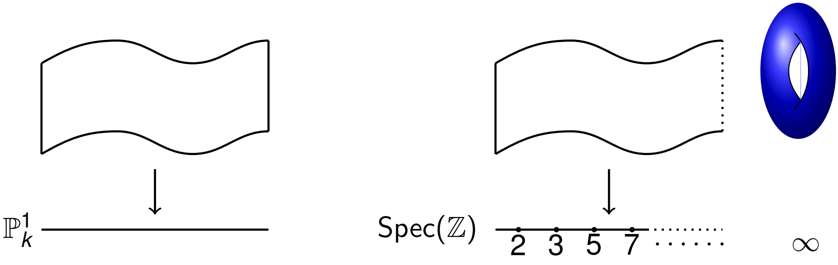

Algebraic geometry

Arakelov geometry

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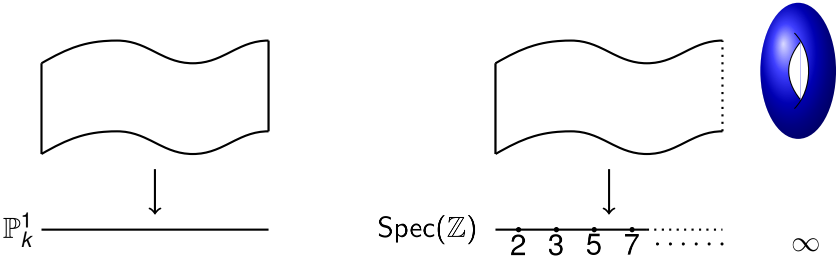

Algebraic geometry Arakelov geometry

Difficulties

- No algebraic structure on the “compactification”
- Analytic and algebraic methods are of very different nature

Geometry vs. arithmetics: case of vector bundles

$k = \mathbb{F}_q$	?
$k[T]$	$\mathbb{Z}$
$C = \mathbb{P}^1_k$	$\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$
$k(C) = k(T)$	$\mathbb{Q}$
vector bundle E on C	$\mathcal{E} = \mathbb{Z}^n$ with $\\|\cdot\\|$ on $\mathcal{E}_\mathbb{R}$
$\deg(E) \in \mathbb{Z}$	$\widehat{\deg}(\mathcal{E}, \\|\cdot\\|) := -\ln\\|e_1 \wedge \cdots \wedge e_n\\| \in \mathbb{R}$
$H^0(C, E)$	$\widehat{H}^0(\mathcal{E}, \\|\cdot\\|) := \{s \in \mathcal{E} \; : \; \\|s\\| \leqslant 1\}$
$h^0(E) := \log_q \#H^0(C, E) \in \mathbb{Z}$	$\widehat{h}^0(\mathcal{E}, \\|\cdot\\|) := \ln \#H^0(C, E) \in \mathbb{R}$

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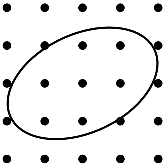
Illustration of the arithmetic case



- $\widehat{H}^0(\mathcal{E}, \|\cdot\|) := \{s \in \mathcal{E} : \|s\| \leqslant 1\}$ is not stable under addition.

Example: Minkowski's first theorem (a particular case)

Theorem

Let $(\mathcal{E}, \|\cdot\|)$ be an Euclidean lattice of rank n.
If $\widehat{\deg}(\mathcal{E}, \|\cdot\|) \geqslant \frac{n}{2}\ln(n)$, then $\widehat{h}^0(\mathcal{E}, \|\cdot\|) > 0$.

Geometric analogue

Let C be a regular projective curve over a field k and E be a vector bundle of rank n on C. If $\deg(E) > n(g(C) - 1)$, then $h^0(E) > 0$.

Adèles

Claude Chevalley

André Weil

Adèles

Claude Chevalley

André Weil

- View function fields and number fields in a unified way
- Treat equitably all places of a global field

Geometry vs. arithmetics: an adelic view

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$\Omega_{k(C)} = \{\text{closed points of } C\}$	$\Omega_{\mathbb{Q}} := \{2, 3, 5, 7, \ldots\} \cup \{\infty\}$
$\forall x \in \Omega_{k(C)}$, $\lVert\cdot\rVert_x$ on $E \otimes_{\mathcal{O}_C} k(C)_x$	$\forall$ prime number p, $\lVert\cdot\rVert_p$ on $\mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Q}_p$
$\lVert s \rVert_x := \inf\{\lvert a \rvert \;:\; a \in k(C)_x^{\times},\, a^{-1}s \in E \otimes_{\mathcal{O}_C} \mathcal{O}_{k(C)_x}\}$	$\lVert s \rVert_p := \inf\{\lvert a \rvert \;:\; a \in \mathbb{Q}_p^{\times},\, a^{-1}s \in \mathcal{E} \otimes_{\mathbb{Z}} \mathbb{Z}_p\}$
$H^0(C, E) = \{s \in E_{k(C)} \;:\; \sup_{x \in \Omega_{k(C)}} \lVert s \rVert_x \leqslant 1\}$	$\widehat{H}^0(\mathcal{E}, \lVert\cdot\rVert) = \{s \in E_{\mathbb{Q}} \;:\; \sup_{\omega \in \Omega_{\mathbb{Q}}} \lVert s \rVert_\omega \leqslant 1\}$

Adelic approach in Arakelov geometry: some references

Adelic vector bundles on a global field

- É. Gaudron, Pentes des fibrés vectoriels adéliques sur un corps global, *Rend. Sem. Mat. Univ. Padova*, **119** (2008), see also Cours de 3e cycle de J.-B. Bost at IHP.

Height of arithmetic variety wrt adelic line bundles

- S. Zhang, Positive line bundles on arithmetic varieties, *J. Amer. Math. Soc.*, **8** (1995).
- R. Rumely, C. Lau, R. Varley, Existence of the sectional capacity, *Mem. Amer. Math. Soc.* **145** (2000).
- A. Moriwaki, Adelic divisors on arithmetic varieties, *Mem. Amer. Math. Soc* **242** (2016).

Toric case

- V. Maillot, Géométrie d'Arakelov des variétés toriques et fibrés intégrables, *Mem. de la SMF* **80** (2000).
- J. Burgos, P. Philippon, M. Sombra, Arithmetic geometry of toric varieties, *Astérisque* **360** (2014).

Beyond the classic adelic setting

Finitely generated field over $\mathbb{Q}$

- A. Moriwaki, Arithmetic height functions over finitely generated fields, *Invent. Math.* **140** (2000).
  - K finitely generated extension of $\mathbb{Q}$
  - B normal projective $\mathbb{Z}$-scheme s.t. $K = \mathrm{Rat}(B)$
  - $\overline{H}_1, \ldots, \overline{H}_d$: C^∞-hermitian line bundles on B, with $d = \dim(B) - 1$.
  - To any projective morphism $\pi : \mathscr{X} \to B$ and hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$ one associates a height function

$$h^{(B,\overline{H}_1,\ldots,\overline{H}_d)}_{(\mathscr{X},\overline{\mathscr{L}})} : \mathscr{X}_K(K^{\mathrm{ac}}) \to \mathbb{R}$$

$$P \mapsto \frac{\widehat{\deg}(\widehat{c}_1(\overline{\mathscr{L}}|_{\Delta_P}) \cdot \widehat{c}_1(\pi^*(\overline{H}_1)|_{\Delta_P}) \cdots \widehat{c}_1(\pi^*(\overline{H}_d)|_{\Delta_P}))}{[K(P):K]},$$

Δ_P being the Zariski closure of $P : \operatorname{Spec} \overline{K} \to \mathscr{X}_K \hookrightarrow \mathscr{X}$.

Beyond the classic adelic setting

Infinite algebraic extensions of $\mathbb{Q}$

- É. Gaudron & G. Rémond, Corps de Siegel, *Crell's journal* **726** (2017)
  - K: algebraic extension of $\mathbb{Q}$
  - $\Omega_K = \varprojlim_{\substack{K/K'/\mathbb{Q} \\ [K':\mathbb{Q}]<\infty}} \Omega_{K'}$, with $\Omega_{K'} = \{\text{places of } K'\}$.
  - Adelic space: K-vector space of finite type E equipped with $(\|\cdot\|_{E,v})_{v\in\Omega_K}$, $\|\cdot\|_{E,v}$ being a norm on $E \otimes_K K_v$.
  - Rigidity: Any basis of E is orthonormal for any v outside of a compact set.
  - Natural questions of Diophantine nature for rigid adelic spaces: Northcott property, Minkowski's theorems, Siegel's lemma etc.

Beyond the classic adelic setting

$\mathbb{R}$-filtrations

- H. Chen, Convergence des polygones de Harder-Narasimhan, *Mem. de la SMF* **120** (2010)
  - Let K be a field and E be a K-vector space of finite type.
  - $\mathbb{R}$-filtration on E: a family $\mathcal{F} = (\mathcal{F}^t(E))_{t\in\mathbb{R}}$ of vector subspaces of E
    - ◇ $t_1 \leqslant t_2 \Rightarrow \mathcal{F}^{t_1}(E) \supset \mathcal{F}^{t_2}(E)$
    - ◇ $\mathcal{F}^t(E) = E$ for t sufficiently negative
    - ◇ $\mathcal{F}^t(E) = \{0\}$ for t sufficiently positive
    - ◇ the function $t \mapsto \mathrm{rk}(\mathcal{F}^t(E))$ is left continuous
  - We equip K with the trivial valuation ($|a| = 1$ for $a \in K^\times$)

$$\{\mathbb{R}\text{-filtrations on } E\} \longleftrightarrow \{\text{ultrametric norms on } E\}$$
$$\mathcal{F} \longleftrightarrow (s \in E) \mapsto \|s\| = \exp(-\sup\{t \in \mathbb{R} \,:\, s \in \mathcal{F}^t(E)\}).$$

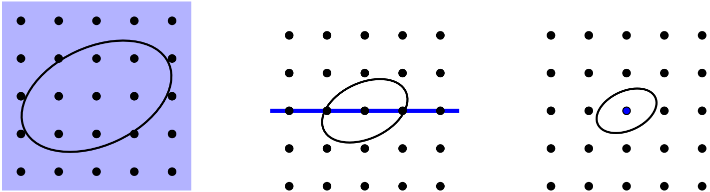
  - Application: projection to trivial valuation arithmetic

Illustration: $\mathbb{R}$-filtration of a lattice

- $\overline{E} = (E, \|\cdot\|)$ an Euclidean lattice

The filtration by minima

$$\mathcal{F}^t(E_{\mathbb{Q}}) := \mathrm{Vect}_{\mathbb{Q}}(\{s \in E \,:\, \|s\| \leqslant \mathrm{e}^{-t}\}), \quad t \in \mathbb{R}.$$



- Captures the logarithmic successive minima.

Adelic curve

- K: a field, M_K: set of all absolute values on K

Definition

We call adelic structure on K a measure space $(\Omega, \mathcal{A}, \nu)$ with a map $\phi : \Omega \to M_K$, $(\omega \in \Omega) \mapsto |\cdot|_\omega$ such that, for any $a \in K^\times$, the function $\omega \mapsto \ln |a|_\omega$ is ν-integrable. $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ is called an adelic curve. If the "product formula"

$$\forall\, a \in K^\times, \quad \int_\Omega \ln |a|_\omega \, \nu(\mathrm{d}\omega) = 0$$

holds, S is said to be proper.

Remark

- Similar to the notion of *M*-field of Gubler.
  W. Gubler, Heights of subvarieties over *M*-fields, in Sympos. Math. XXXVII, 1997.
- Our purpose is to develop a geometry of vector bundles on adelic curves to study arithmetic invariants of linear series.

Adelic vector bundles: dominancy

- Fix an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ and a vector space of finite type E over K.
- For $\omega \in \Omega$, let K_ω be the completion of K wrt $|\cdot|_\omega$.

Definition

- We call norm family on E any family $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$, where $\|\cdot\|_\omega$ is a norm on $E \otimes_K K_\omega$. ξ is said to be hermitian if $\|\cdot\|_\omega$ is ultrametric (resp. euclidean/hermitian) once $|\cdot|_\omega$ is non-archimedean (resp. archimedean).
- ξ induces a dual norm family $\xi^\vee = (\|\cdot\|_\omega^*)_{\omega\in\Omega}$ on $E^\vee$.
- We say that ξ is upper dominated if for any $s \in E \setminus \{0\}$ the function $(\omega \in \Omega) \mapsto \ln\|s\|_\omega$ is bounded from above by a ν-integrable function.
- We say that ξ is dominated if both ξ and $\xi^\vee$ are upper dominated.
- If ξ is dominated, for any $s \in E \setminus \{0\}$, $(\omega \in \Omega) \mapsto |\ln\|s\|_\omega|$ is bounded from above by a ν-integrable function.

Adelic vector bundles: measurability

- Fix an adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$, a vector space of finite type E over K and a norm family $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$.

Definition

- We say that ξ is measurable if for any $s \in E$ the function $(\omega \in \Omega) \mapsto \|s\|_\omega$ is $\mathcal{A}$-measurable.
- If ξ is dominated and if ξ and $\xi^\vee$ are both measurable, we say that (E, ξ) is an adelic vector bundle on S. If in addition $\mathrm{rk}_K(E) = 1$, we say that (E, ξ) is an adelic line bundle.
- For $s \in E \setminus \{0\}$ the Arakelov degree of s wrt ξ is defined as

$$\widehat{\deg}_\xi(s) := -\int_{\omega\in\Omega} \ln\|s\|_\omega \,\nu(\mathrm{d}\omega), \quad s \in E \setminus \{0\}.$$

- If S is proper, then for any $a \in K^\times$, $\widehat{\deg}_\xi(as) = \widehat{\deg}_\xi(s)$.

Examples of proper adelic curves

Number fields

- K: number field, Ω_K: set of places, $\mathcal{A}$: discrete σ-algebra
- for $\omega \in \Omega_K$, $\nu(\{\omega\}) = [K_\omega : \mathbb{Q}_\omega]$.

One trivial valuation

- $\Omega = \{\omega\}$, $\mathcal{A}$ is the trivial σ-algebra, $\nu(\{\omega\}) = 1$.
- $\mathbb{R}$-filtered vector space of finite rank $(E, \mathcal{F})$ can be viewed as an adelic vector bundle $(E, \xi_{\mathcal{F}})$

$$\forall\, s \in E \setminus \{0\}, \quad \widehat{\deg}_{\xi_{\mathcal{F}}}(s) = \sup\{t \in \mathbb{R} \,:\, s \in \mathcal{F}^t(E)\}.$$

Several copies of the trivial valuation

- $(\Omega, \mathcal{A}, \nu)$ arbitrary measure space
- $\phi : \Omega \to M_K$, $\omega \mapsto$ the trivial absolute value

Example: function field over $\mathbb{Q}$

$K = \mathbb{Q}(T)$ field of rational functions of one variable T

Three types of absolute values

- For closed $x \in \mathbb{P}^1_{\mathbb{Q}}$, let $\mathrm{ord}_x(\cdot)$ be valuation on K of x.
  - If $x \in \mathbb{P}^1_{\mathbb{Q}} \setminus \{\infty\} = \mathbb{A}^1_{\mathbb{Q}}$, it corresponds to an irreducible $F_x \in \mathbb{Z}[T]$ with coprime coefficients and we let

    $$\forall\, \varphi \in K, \quad |\varphi|_x := Ma(x)^{-\,\mathrm{ord}_x(\varphi)},$$

    where $Ma(x) := \exp(\int_0^1 \ln|F_x(e^{2\pi i t})|dt)$ is the Mahler measure of F_x.
  - By convention $|\cdot|_\infty$ is the trivial absolute value on K.
- For prime number p, let $|\cdot|_p$ be the p-adic absolute value on $\mathbb{Q}$, which extends naturally to $\mathbb{Q}(T)$:

  $$\forall\, f = a_d T^d + \cdots + a_0 \in \mathbb{Q}[T], \quad |f|_p := \max_{j \in \{0,\ldots,d\}} |a_j|_p.$$

- For $t \in [0,1]$ with $e^{2\pi i t}$ transcendental, let $|\varphi|_t := |\varphi(e^{2\pi i t})|$.

Function field on $\mathbb{Q}$: adelic structure

- $\Omega_h = \{\text{closed points of } \mathbb{P}^1_{\mathbb{Q}}\}$.
- $\mathcal{P} = \{\text{prime numbers}\}$.
- $[0,1]_* = \{t \in [0,1] \;:\; e^{2\pi i t} \text{ is transcendental}\}$.

Adelic structure on $\mathbb{Q}(T)$

- $\Omega = \Omega_h \amalg \mathcal{P} \amalg [0,1]_*$.
- Let $\mathcal{A}$ be the σ-algebra on Ω generated by the discrete σ-algebras on Ω_h and $\mathcal{P}$, and the Borel σ-algebra on $[0,1]_*$.
- Let ν be the measure on Ω such that $\nu(\{\omega\}) = 1$ for $\omega \in \Omega_h \cup \mathcal{P}$ and that $\nu|_{[0,1]_*} =$ the Lebesgue measure.

Product formula

For $f = aF_{x_1}^{r_1} \cdots F_{x_n}^{r_n} \in \mathbb{Q}[T]$, with $a \in \mathbb{Q}^\times$, one has

$$\int_0^t \ln|f(e^{2\pi i t})|dt = \ln|a| + \sum_{j=1}^n r_j \ln Ma(x_j).$$

Other examples

Polarised variety

- Let k be a field and X be a normal projective scheme of dimension $d \geqslant 1$ over $\operatorname{Spec} k$.
- $K = k(X)$ field of rational functions.
- We fix $D_1, \ldots, D_{d-1}$ ample Cartier divisors on X.
- $\Omega = X^{(1)} = \{\text{prime divisors in } X\}$, equipped with the discrete σ-algebra.
- For $Y \in X^{(1)}$, $\nu(\{Y\}) := \deg(D_1 \cdots D_{d-1} \cap [Y])$.

Polarised arithmetic variety

- generalising the case of $\mathbb{Q}(T)$
- Ω contains a horizontal part, a vertical part over finite places of $\mathbb{Q}$, and a vertical part over ∞.

Algebraic covering

Fix $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ and an algebraic extension L/K.

Construction of an adelic structure on L

- $\Omega_L := \Omega \times_{M_K,\phi} M_L$
- Let $\pi_{L/K} : \Omega_L \to \Omega$ and $\phi_L : \Omega_L \to M_L$, $(x \in \Omega_L) \mapsto |\cdot|_x$ be the projection maps.
- Let $\mathcal{A}_L$ be the smallest σ-algebra making $\pi_{L/K}$ and the functions $(x \in \Omega_L) \mapsto |a|_x$ measuralbe, where $a \in L$.

Theorem

There exists a unique measure ν_L on $(\Omega_L, \mathcal{A}_L)$ and a unique continuous linear map $I_{L/K} : \mathcal{L}^1(\Omega_L, \mathcal{A}_L, \nu_L) \to \mathcal{L}^1(\Omega, \mathcal{A}, \nu)$ s.t.

- For any $f \in \mathcal{L}^1(\Omega_L, \mathcal{A}_L, \nu_L)$, $\int_{\Omega_L} f\, d\nu_L = \int_\Omega I_{L/K}(f)\, d\nu$.
- For any $g \in \mathcal{L}^1(\Omega, \mathcal{A}, \nu)$, $\int_\Omega g\, d\nu = \int_{\Omega_L} g \circ \pi_{L/K}\, d\nu_L$.
- For any K'/K finite sub-extension of L/K and any $a \in K'$, $[K' : K] \int_{\Omega_L} \ln |a|_x \, \nu_L(dx) = \int_\Omega \ln |N_{K'/K}(a)|_\omega \, \nu(d\omega)$.

Geometry of numbers of an adelic curve

Difficulties

- Not adequate to consider integral models.
- Northcott property is not true in general.
- The unit adelic ball may contain infinitely many elements of the underlying field.
- Minkowski's theorem does not apply to general adelic curves (counter-example: 3 copies of the trivial valuation).

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Idea: develop a slope theory for adelic vector bundles

- slope theory of Hermitian vector bundles in the case of number fields:
  J.-B. Bost, Appendix of Bourbaki talk "Périodes et isogénies des variétés abéliennes sur les corps de nombres", *Astérisque* **237** (1996).

Slope theory on an adelic curve

- Fix a proper adelic curve $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$.
- Assume that there exist a countable subfield $K_0 \subset K$ which is dense in every K_ω ($\omega \in \Omega$).
- The category of adelic vector bundles on S is stable by usual algebraic operations (subbundle, quotient, direct sum, tensor product, determinant etc)

Arakelov degree and slope

Let (E, ξ) be an adelic vector bundle on S, $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$. The Arakelov degree of (E, ξ) is defined as

$$\widehat{\deg}(E, \xi) := -\int_\Omega \ln\|s_1 \wedge \cdots \wedge s_d\|_{\omega,\det}\,\nu(\mathrm{d}\omega),$$

where $(s_i)_{i=1}^d$ is a basis of E over K.

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where $(s_i)_{i=1}^d$ is a basis of E over K.
If $d > 0$, the slope of (E, ξ) is defined as

$$\widehat{\mu}(E, \xi) := \frac{1}{d}\widehat{\deg}(E, \xi).$$

Minimal slope and $\mathbb{R}$-filtration by minimal slope

Minimal slope

Let $\overline{E} = (E, \xi_E)$ be a non-zero adelic vector bundle. The minimal slope of (E, ξ_E) is defined as

$$\widehat{\mu}_{\min}(\overline{E}) := \inf_{E \twoheadrightarrow G \neq \{0\}} \widehat{\mu}(\overline{G}),$$

where on each quotient space G we consider the family of quotient norms.

$\mathbb{R}$-filtration by minimal slope

$$\forall\, t \in \mathbb{R}, \quad \mathcal{F}_{\mathrm{hn}}^{t}(\overline{E}) := \sum_{\substack{0 \neq F \subset E \\ \widehat{\mu}_{\min}(\overline{F}) \geqslant t}} F.$$

$\overline{E}$ is said to be semistable if $\mathcal{F}_{\mathrm{hn}}$ only has one jump, or equivalently $\widehat{\mu}_{\min}(\overline{F}) \leqslant \widehat{\mu}_{\min}(\overline{E})$ for any non-zero subspace $F \subset E$.

Modified slope and Harder-Narasimhan filtration

$\overline{E} = (E, \xi)$ non-zero adelic vector bundle on S, $\xi = (\|\cdot\|_\omega)_{\omega\in\Omega}$.

Modified degree and slope

Let $\widetilde{\deg}(\overline{E}) := -\int_{\mathbb{R}} t\, d\, \mathrm{rk}_K(\mathcal{F}^t_{\mathrm{hn}}(\overline{E}))$ and $\widetilde{\mu}(\overline{E}) := \widetilde{\deg}(\overline{E})/\,\mathrm{rk}(E)$.

- $\widetilde{\deg}(\overline{E})$ is just the weighted sum of jump points of the $\mathbb{R}$-filtration $\mathcal{F}_{\mathrm{hn}}$.
- If ξ is hermitian, then $\widetilde{\deg}(\overline{E}) = \widehat{\deg}(\overline{E})$.

Theorem

There exists a unique flag $0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_n = E$ of vector subspaces of E such that each subquotient $\overline{E_i/E_{i-1}}$ is semistable and that $\widetilde{\mu}(\overline{E_1/E_0}) > \ldots > \widetilde{\mu}(\overline{E_n/E_{n-1}})$.

Modified slope and Harder-Narasimhan filtration

$\overline{E} = (E, \xi)$ non-zero adelic vector bundle on S, $\xi = (\|\cdot\|_\omega)_{\omega \in \Omega}$.

Modified degree and slope

Let $\widetilde{\deg}(\overline{E}) := -\int_{\mathbb{R}} t \, d \operatorname{rk}_K(\mathcal{F}_{\mathrm{hn}}^t(\overline{E}))$ and $\widetilde{\mu}(\overline{E}) := \widetilde{\deg}(\overline{E}) / \operatorname{rk}(E)$.

- $\widetilde{\deg}(\overline{E})$ is just the weighted sum of jump points of the $\mathbb{R}$-filtration $\mathcal{F}_{\mathrm{hn}}$.
- If ξ is hermitian, then $\widetilde{\deg}(\overline{E}) = \widehat{\deg}(\overline{E})$.

Theorem

There exists a unique flag $0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_n = E$ of vector subspaces of E such that each subquotient $\overline{E_i/E_{i-1}}$ is semistable and that $\widetilde{\mu}(\overline{E_1/E_0}) > \ldots > \widetilde{\mu}(\overline{E_n/E_{n-1}})$.

Remark

New even for lattices (of non-Euclidean norm), compared with

- U. Stuhler, Ein Bemerkung zur Reduktionstheorie quadratischen Formen, *Archiv der Math.* **27** (1976).

Riemann-Roch theorem on adelic curves

Candidate for replacing $\widehat{h}^0$

For any adelic vector bundle $\overline{E}$ on S, let

$$\widehat{\deg}_+(\overline{E}) := \sup_{\{0\}\neq F\subset E} \widehat{\deg}(\overline{F}).$$

Theorem

Let $\overline{E}$ be an adelic vector bundle on S. One has

$$0 \leqslant \widehat{\deg}(\overline{E}) - (\widehat{\deg}_+(\overline{E}^{\vee\vee}) - \widehat{\deg}_+(\overline{E}^{\vee})) \leqslant \frac{1}{2}\ln(\mathrm{rk}(E))\nu(\Omega_\infty),$$

where $\Omega_\infty = \{\omega \in \Omega \,:\, |\cdot|_\omega \text{ is archimedean}\}$.

Remark

$\overline{E}^{\vee\vee} = \overline{E}$ once $\|\cdot\|_\omega$ is ultrametric for any $\omega \in \Omega \setminus \Omega_\infty$.

Higher dimensional geometry

Let $\pi : X \to \operatorname{Spec} K$ be an integral projective K-scheme and L be a line bundle on X.

Metric family

By metric family on L, we refer to a family $\varphi = (\varphi_\omega)_{\omega \in \Omega}$ where φ_ω is a continuous metric on L_ω^{an}.

- For defining an adelic metric, we can not follow the classic approach since no integral model can be considered.

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Dominancy

Assume that L is very ample. We equip $E = H^0(X, L)$ with a hermitian norm family ξ such that (E, ξ) is an adelic vector bundle. Let φ_ξ be the family of Fubini-Study metrics on L.

- We say that φ is dominated if the function

$$(\omega \in \Omega) \mapsto \sup_{x \in X_\omega^{\mathrm{an}}} \left| \ln \frac{|\cdot|_{\varphi_\omega}(x)}{|\cdot|_{\varphi_{\xi,\omega}}(x)} \right|$$

is bounded from above by a ν-integrable function.

Higher dimensional geometry

Measurability

We say that a metric family φ is measurable if for any closed point P of X the norm family $P^*(\varphi)$ on $P^*(L)$ is measurable.

Adelic line bundle

Let L be a line bundle on X and φ be a metric family on X. If there exist very ample line bundles L_1 and L_2 equipped with dominated and measurable metric families φ_1 and φ_2 such that $L \cong L_1 \otimes L_2^\vee$ and $\varphi \cong \varphi_1 \otimes \varphi_2^\vee$, we say that (L, φ) is an adelic line bundle on X.

- Independent of various choices.
- Stable under tensor product and dual.
- $\widehat{\mathrm{Pic}}(X)$: group of isomorphism classes of adelic vector bundles on X

Volume function

Linear system

Let (L, φ) be adelic line bundle. Then $(H^0(X, L), (\|\cdot\|_{\varphi_\omega})_{\omega\in\Omega})$ is an adelic vector bundle on S, denoted by $\pi_*(L, \varphi)$, where

$$\|s\|_{\varphi_\omega} := \sup_{x\in X_\omega^{\mathrm{an}}} |s|_{\varphi_\omega}(x).$$

Volume function

Let (L, φ) be an adelic line bundle on X. The volume of (L, φ) is define as

$$\widehat{\mathrm{vol}}(L, \varphi) := \limsup_{n\to+\infty} \frac{\widehat{\deg}_+(\pi_*((L, \varphi)^{\otimes n}))}{n^{d+1}/(d+1)!}.$$

If $\widehat{\mathrm{vol}}(L, \varphi) > 0$, (L, φ) is said to be big.

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- lim sup can be replaced by lim
- convex geometry interpretation
- Brunn-Minkowski inequality
- continuity of volume function

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Graduate course

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Further research topics and problems

Fundamental problems

- For a fixed field K, what are proper adelic curve structures on the field?
- Existence of compactification for a non-proper adelic curve.
- Categorical study of adelic curve and adelic vector bundles.

Geometric problems

- Relation between the volume function and Gubler's height (Hilbert-Samuel type theorem).
- Arithmetic intersection theory for projective varieties over an adelic curve. Riemann-Roch.
- Differentiability of the volume function, Bogomolov type of problems, algebraic dynamical system over general fields.