Hypoelliptic Laplacian and the trace formula

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Poisson's formula

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$$+\sum_{\gamma\neq 0}\frac{\operatorname{Vol}_\gamma}{\sqrt{2\pi t}}\frac{\exp\left(-\ell_\gamma^2/2t - t/8\right)}{2\sinh\left(\ell_\gamma/2\right)}.$$

- Sum of orbital integrals.

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- . . . that connects Selberg's trace formula with index theory, and Riemann-Roch.

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- $s \to 0$ gives Chern-Gauss-Bonnet $\chi(X) = \int_X e(TX)$.

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- $$L(g)\,|_{\mathrm{global}_{s=+\infty}} \xrightarrow{\mathrm{Tr_s}^{\Omega^{\cdot}(X,\mathbf{R})}\left[g \exp\left(-sD^{X,2}\right)\right]} \int_{X_g} e(TX_g)\,|_{\mathrm{local}_{s=0}}.$$

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5. Yes!

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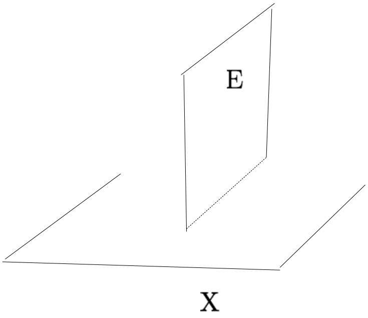
- Its cohomology is equal to $C^\infty(X, \mathbf{R})$.

A picture



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- …and $N^{\Lambda^\cdot(E^*)}$ number operator on $\Lambda^\cdot(E^*)$.
- Via Bargmann isomorphism, we get instead $\left(S^\cdot(E^*) \otimes \Lambda^\cdot(E^*), d^E\right)$ algebraic de Rham complex.

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- Here E will be $TX \oplus N$.

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- Kostant found a different Dirac operator D^{Ko} adapted to reductive Lie groups.

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Remark

$\widehat{D}^{\mathrm{Ko}}$ acts on $C^\infty(G, \Lambda^\cdot(\mathfrak{g}^*))$, $C^{\mathfrak{g}}$ acts on $C^\infty(G, \mathbf{R})$.

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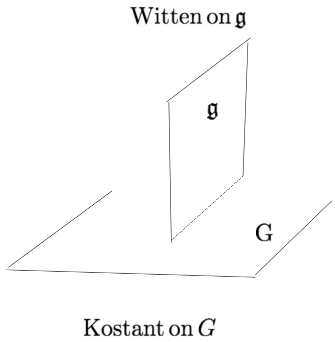

Then quotient by K.

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$+ \frac{\sqrt{-1}}{b}\left(-d^{\mathfrak{k}} - Y^{\mathfrak{k}} \wedge + d^{\mathfrak{k}*} + i_{Y^{\mathfrak{k}}}\right)$.
- Parameter b related to spectral sequence.

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Example

$G = \mathrm{SL}_2(\mathbf{R})$, $K = S^1$, X upper half-plane, $TX \oplus N$ of dimension 3.

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- $\mathcal{L}_b^X = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},X,2} + \mathfrak{D}_b^{X,2}\right)$ acts on

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Remark

Using fiberwise Bargmann isomorphism, $\mathcal{L}_b^X$ acts on

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$$\mathcal{L}_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^{\cdot}(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\Bigg(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\,\mathrm{ad}\left(Y^N\right)\right)\Bigg).$$

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$b \to +\infty$, geod. flow $\nabla_{Y^{TX}}$ dominates $\Rightarrow$ closed geodesics.

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The case of locally symmetric spaces

- Introduce extra twisting by representation of K on E.
- $\Gamma \subset G$ cocompact torsion free.
- $Z = \Gamma \setminus X$ compact locally symmetric.

A fundamental identity

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Theorem

For $t > 0, b > 0$,

$$\mathrm{Tr}^{C^{\infty}(Z,E)}\left[\exp\left(-t\left(C^{Z}-c\right)/2\right)\right] = \mathrm{Tr}_{\mathrm{s}}\left[\exp\left(-t\mathcal{L}_{b}^{Z}\right)\right].$$

Splitting the identity

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2. The identity splits as identity of orbital integrals.

Semisimple orbital integrals

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- $\gamma \in G$ semisimple, $[\gamma]$ conjugacy class.
- For $t > 0$, $\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t\left(C^X - c\right)/2\right)\right]$ orbital integral of heat kernel on orbit of γ:

$$I\left([\gamma]\right) = \int_{Z(\gamma)\backslash G} \mathrm{Tr}^E\left[p_t^X\left(g^{-1}\gamma g\right)\right] dg.$$

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•$X(\gamma) \subset X$ totally geodesic symmetric space for the centralizer $Z(\gamma)$.

Geometric description of the orbital integral

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$$I(\gamma) = \int_{N_{X(\gamma)/X}} \operatorname{Tr}\left[\gamma p_t^X(Y, \gamma Y)\right] \underbrace{r(Y)}_{\text{Jacobian}} dY.$$

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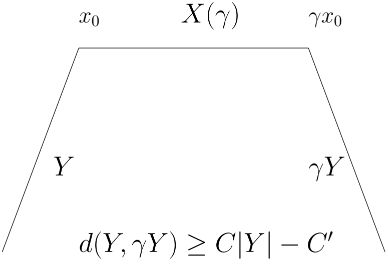

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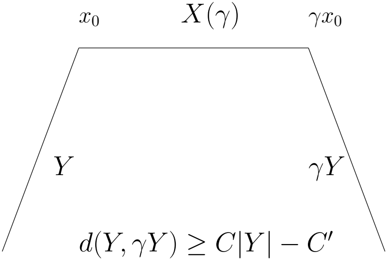

$$p_t^X\left(x, x'\right) \leq C \exp(-C' d^2(x, x')).$$

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Remark

The proof uses the fact that $\mathrm{Tr}^{[\gamma]}$ is a trace on the algebra of G-invariants smooth kernels on X with Gaussian decay.

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$$\int_{\mathfrak{k}(\gamma)} J_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1}\right)\exp\left(-i\rho^E\left(Y_0^{\mathfrak{k}}\right)\right)\right]$$

$$\exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

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Note the integral on $\mathfrak{k}(\gamma)$...

The function $J_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$

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Definition

$$J_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1-\mathrm{Ad}\left(\gamma\right)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1-\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\right.$$

$$\left.\frac{\det\left(1-\exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

Analogy with Atiyah-Bott

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- Here TX replaced by $TX \ominus N$.

The D and R genera

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- $p_t(x)$ heat kernel on S^1.
- $D(x) = -\int_0^{+\infty} \mathrm{Log} r \frac{\partial}{\partial x}[p_r](rx)\, dr$.
- With Soulé, we proved that

$$D(x) = \sum_{\substack{n \geq 1 \\ n \,\mathrm{odd}}} \left\{ \Gamma'(1) + \sum_{j=1}^{n} \frac{1}{j} + 2\frac{\zeta'(-n)}{\zeta(-n)} \right\} \zeta(-n) \frac{x^n}{n!}.$$

Connection with the R genus

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- $D(x) = R(x) + \Gamma'(1)\frac{\widehat{A}'}{\widehat{A}}(x)$.
- Above construction played important role in proof with Lebeau of embedding formulas.

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- I obtained the function $D(x)$ as part of a construction of characteristic classes on holomorphic vector bundles, in which the fiberwise harmonic oscillator plays a crucial role, as well as ideas of infinite dimensional intersection theory.
- Fiberwise Witten Laplacian replaced by its complex version.
- The harmonic oscillator plays a very similar role to what it does for orbital integral.

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