Enriques $2n$-folds and Analytic Torsion

Intercity Seminar on Arakelov Geometry, Copenhagen 2018

Ken-Ichi Yoshikawa

Kyoto University

September 5, 2018

Contents

1. A theorem of Borcherds
2. Simple Enriques $2n$-folds
3. Analytic torsion of Calabi-Yau $2n$-folds and hyperkähler manifolds
4. Holomorphic torsion invariant of Enriques $2n$-folds
5. Quasi-affinity of the moduli space
6. Automorphy of τ_{Enr} : hyperkähler type case
7. Simple Enriques $2n$-folds associated with $(2, \ldots, 2)$-complete intersections
8. Simple Enriques $2n$-folds associated with Enriques surfaces

A theorem of Borcherds

In 1996, Borcherds proved the following

Theorem (Borcherds)

The moduli space of Enriques surfaces is quasi-affine.

A theorem of Borcherds

In 1996, Borcherds proved the following

Theorem (Borcherds)

The moduli space of Enriques surfaces is quasi-affine.

Corollary

Every family of Enriques surfaces without singular fibers over a compact connected complex manifold is isotrivial, i.e., all fibers are isomorphic to each other.

A theorem of Borcherds

In 1996, Borcherds proved the following

Theorem (Borcherds)

The moduli space of Enriques surfaces is quasi-affine.

Corollary

Every family of Enriques surfaces without singular fibers over a compact connected complex manifold is isotrivial, i.e., all fibers are isomorphic to each other.

Borcherds proved the theorem by constructing an automorphic form Φ nowhere vanishing on the moduli space. This remarkable automorphic form Φ is called the Borcherds Φ-function or Borcherds-Enriques form.

It is possible to construct the Borcherds Φ-function from analytic torsion of Enriques surfaces.

Goal of talk

- Construction of a holomorphic torsion invariant of higher dimensional analogues of Enriques surfaces
- isotriviality of smooth family of those manifolds over compact base (quasi-affinity of the moduli space under some assumption)
- automorphy of the invariant in certain cases
- explicit formula as a (automorphic) function on the moduli space

Bogomolov decomposition theorem

Definition

X : compact connected Kähler manifold

- X is Calabi-Yau $\iff K_X \cong \mathcal{O}_X$ and $h^q(\mathcal{O}_X) = 0$ $(0 < q < \dim X)$.
- X is hyperkähler $\iff \pi_1(X) = \{1\}$ and $H^0(\Omega_X^2)$ is generated by a holomorphic symplectic form.

Bogomolov decomposition theorem

Definition

X : compact connected Kähler manifold

- X is Calabi-Yau $\iff K_X \cong \mathcal{O}_X$ and $h^q(\mathcal{O}_X) = 0$ $(0 < q < \dim X)$.
- X is hyperkähler $\iff \pi_1(X) = \{1\}$ and $H^0(\Omega^2_X)$ is generated by a holomorphic symplectic form.

Theorem (Bogomolov)

Let Y be a compact Kähler manifold with torsion canonical bundle. Then there is an étale finite covering $X \to Y$ such that

$$X \cong T \times \prod_i U_i \times \prod_j S_j$$

T : complex torus, U_i : simply connected Calabi-Yau, S_j : hyperkähler

Simple Enriques $2n$-folds

Definition

A compact Kähler manifold Y is *simple Enriques* of *index* $d \in \mathbb{Z}_{>1}$ if

(1) $K_Y \not\cong \mathcal{O}_Y$ and $K_Y^{\otimes d} \cong \mathcal{O}_Y$.

(2) $\chi(\mathcal{O}_Y) > 0$.

(3) universal covering of Y is irreducible, i.e., Bogomolov decomposition of Y consists of a unique factor.

Simple Enriques $2n$-folds

Definition

A compact Kähler manifold Y is *simple Enriques* of *index* $d \in \mathbb{Z}_{>1}$ if

(1) $K_Y \not\cong \mathcal{O}_Y$ and $K_Y^{\otimes d} \cong \mathcal{O}_Y$.

(2) $\chi(\mathcal{O}_Y) > 0$.

(3) universal covering of Y is irreducible, i.e., Bogomolov decomposition of Y consists of a unique factor.

Remark 1. Closely related notions were introduced by two groups:

- Boissière-Nieper-Weisskirchen-Sarti (2011): *Enriques variety*
- Oguiso-Schröer (2011): *Enriques manifold*

Simple Enriques $\Longleftrightarrow$ irreducible weak Enriques in the sense of BNWS

Fact (Boissière-Nieper-Weisskirchen-Sarti, Oguiso-Schröer)

Y : *simple Enriques with* $K_Y^{\otimes d} \cong \mathcal{O}_Y$ $\qquad \widetilde{Y}$: *universal covering of* Y

(1) $\widetilde{Y}$ *is either Calabi-Yau of even dimension or hyperkähler.*

(2) $\pi_1(Y)$ *is a finite cyclic group of order* $d > 1$.

(3) $\widetilde{Y}$: *Calabi-Yau* $\Longrightarrow d = 2$.

(4) $\widetilde{Y}$: *hyperkähler* $\Longrightarrow d | (n+1)$.

Fact (Boissière-Nieper-Weisskirchen-Sarti, Oguiso-Schröer)

Y : simple Enriques with $K_Y^{\otimes d} \cong \mathcal{O}_Y$ $\widetilde{Y}$: universal covering of Y

(1) *$\widetilde{Y}$ is either Calabi-Yau of even dimension or hyperkähler.*

(2) *$\pi_1(Y)$ is a finite cyclic group of order $d > 1$.*

(3) *$\widetilde{Y}$: Calabi-Yau $\Longrightarrow d = 2$.*

(4) *$\widetilde{Y}$: hyperkähler $\Longrightarrow d|(n+1)$.*

Definition

Y : simple Enriques

- Y : Calabi-Yau type $\Longleftrightarrow$ $\widetilde{Y}$: Calabi-Yau
- Y : hyperkähler type $\Longleftrightarrow$ $\widetilde{Y}$: hyperkähler

Analytic torsion

(X, γ) : compact Kähler manifold
$\zeta_q(s)$: ζ-function of Laplacian $\square_q = (\bar{\partial} + \bar{\partial}^*)^2$ acting on $A^{0,q}(X)$

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \lambda^{-s} \dim E(\square_q, \lambda)$$

Analytic torsion

(X, γ) : compact Kähler manifold
$\zeta_q(s)$: ζ-function of Laplacian $\square_q = (\bar{\partial} + \bar{\partial}^*)^2$ acting on $A^{0,q}(X)$

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \lambda^{-s} \dim E(\square_q, \lambda)$$

Fact

$\zeta_q(s)$ converges absolutely when $\Re s > \dim X$, extends to a meromorphic function on $\mathbb{C}$, holomorphic at $s = 0$

Analytic torsion

(X, γ) : compact Kähler manifold
$\zeta_q(s)$: ζ-function of Laplacian $\square_q = (\bar{\partial} + \bar{\partial}^*)^2$ acting on $A^{0,q}(X)$

$$\zeta_q(s) := \sum_{\lambda \in \sigma(\square_q) \setminus \{0\}} \lambda^{-s} \dim E(\square_q, \lambda)$$

Fact

$\zeta_q(s)$ converges absolutely when $\Re s > \dim X$, extends to a meromorphic function on $\mathbb{C}$, holomorphic at $s = 0$

Definition

Analytic torsion of (X, γ) is defined as

$$\tau(X, \gamma) := \exp\{-\sum_{q \geq 0} (-1)^q q \, \zeta_q'(0)\}$$

Analytic torsion of Calabi-Yau and hyperkähler manifolds

X : Calabi-Yau or hyperkähler of even dimension $2n$
η : nowhere vanishing canonical form on X
γ : Kähler form on X
$c_1(X,\gamma)$: first Chern form of (TX,γ)
$\mathrm{Td}(X,\gamma)$: Todd form of (TX,γ)

X : Calabi-Yau or hyperkähler of even dimension $2n$
η : nowhere vanishing canonical form on X
γ : Kähler form on X
$c_1(X,\gamma)$: first Chern form of (TX,γ)
$\mathrm{Td}(X,\gamma)$: Todd form of (TX,γ)

Theorem

Analytic torsion of (X,γ) is given by

$$\tau(X,\gamma) = \exp\left\{-\frac{1}{2}\int_X \log\left(\frac{\eta\wedge\overline{\eta}}{\gamma^{2n}/(2n)!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|_{L^2}^2}\right)\frac{\mathrm{Td}(X,\gamma)}{\mathrm{Td}(c_1(X,\gamma))}\right\}.$$

In particular, if $\mathrm{Ric}\,\gamma = 0$, then $\tau(X,\gamma) = 1$.

Holomorphic torsion invariant of Enriques $2n$-folds

Y : Enriques $2n$-fold of index d

Ξ : pluricanonical form of weight $d \Longrightarrow |\Xi|^{\frac{2}{d}} := |\Xi \otimes \overline{\Xi}|^{\frac{1}{d}}$ measure on Y

γ : Kähler form on Y

Holomorphic torsion invariant of Enriques $2n$-folds

Y : Enriques $2n$-fold of index d

Ξ : pluricanonical form of weight $d \Longrightarrow |\Xi|^{\frac{2}{d}} := |\Xi \otimes \overline{\Xi}|^{\frac{1}{d}}$ measure on Y

γ : Kähler form on Y

Bott-Chern term

$$A(Y,\gamma) := \exp\left\{\frac{1}{2}\int_Y \log\left(\frac{|\Xi|^{\frac{2}{d}}}{\gamma^{2n}/(2n)!}\cdot\frac{\mathrm{Vol}(Y,\gamma)}{\|\Xi\|_{L^{\frac{2}{d}}}}\right)\frac{\mathrm{Td}(Y,\gamma)}{\mathrm{Td}(c_1(Y,\gamma))}\right\}.$$

If $p\colon \widetilde{Y} \to Y$ is the universal covering, then $A(Y,\gamma) = \tau(\widetilde{Y}, p^*\gamma)^{-1/d}$.

Holomorphic torsion invariant of Enriques $2n$-folds

Y : Enriques $2n$-fold of index d

Ξ : pluricanonical form of weight $d \Longrightarrow |\Xi|^{\frac{2}{d}} := |\Xi \otimes \overline{\Xi}|^{\frac{1}{d}}$ measure on Y

γ : Kähler form on Y

Bott-Chern term

$$A(Y,\gamma) := \exp\left\{\frac{1}{2}\int_Y \log\left(\frac{|\Xi|^{\frac{2}{d}}}{\gamma^{2n}/(2n)!}\cdot\frac{\mathrm{Vol}(Y,\gamma)}{\|\Xi\|_{L^{\frac{2}{d}}}}\right)\frac{\mathrm{Td}(Y,\gamma)}{\mathrm{Td}(c_1(Y,\gamma))}\right\}.$$

If $p\colon \widetilde{Y}\to Y$ is the universal covering, then $A(Y,\gamma)=\tau(\widetilde{Y},p^*\gamma)^{-1/d}$.

Definition

$$\tau_{\mathrm{Enr}}(Y) := \begin{cases} \tau(Y,\gamma)\mathrm{Vol}(Y,\gamma)^{\frac{1}{d}}A(Y,\gamma) & (Y\ :\ \text{Calabi-Yau type}) \\ \tau(Y,\gamma)\mathrm{Vol}(Y,\gamma)^{\frac{(n+1)(d-1)}{2nd}}A(Y,\gamma) & (Y\ :\ \text{hyperkähler type}) \end{cases}$$

Theorem

*$\tau_{\mathrm{Enr}}(Y)$ is independent of the choice of a Kähler metric γ on Y.
In particular, $\tau_{\mathrm{Enr}}(Y)$ is an invariant of Y.*

Theorem

$\tau_{\mathrm{Enr}}(Y)$ is independent of the choice of a Kähler metric γ on Y. In particular, $\tau_{\mathrm{Enr}}(Y)$ is an invariant of Y.

Remark 1. This theorem is a consequence of the anomaly formula for Quillen metrics due to Bismut-Gillet-Soulé.

Theorem

$\tau_{\mathrm{Enr}}(Y)$ is independent of the choice of a Kähler metric γ on Y.
In particular, $\tau_{\mathrm{Enr}}(Y)$ is an invariant of Y.

Remark 1. This theorem is a consequence of the anomaly formula for Quillen metrics due to Bismut-Gillet-Soulé.

Question

τ_{Enr} is a function on the moduli space of simple Enriques $2n$-folds.
What is this function on the moduli space?

Theorem

$\tau_{\mathrm{Enr}}(Y)$ is independent of the choice of a Kähler metric γ on Y. In particular, $\tau_{\mathrm{Enr}}(Y)$ is an invariant of Y.

Remark 1. This theorem is a consequence of the anomaly formula for Quillen metrics due to Bismut-Gillet-Soulé.

Question

τ_{Enr} is a function on the moduli space of simple Enriques $2n$-folds. What is this function on the moduli space?

Remark 2. When $n = 1$, τ_{Enr} is given by the Petersson norm of the Borcherds Φ-function.

Theorem

$\tau_{\mathrm{Enr}}(Y)$ is independent of the choice of a Kähler metric γ on Y.
In particular, $\tau_{\mathrm{Enr}}(Y)$ is an invariant of Y.

Remark 1. This theorem is a consequence of the anomaly formula for Quillen metrics due to Bismut-Gillet-Soulé.

Question

τ_{Enr} is a function on the moduli space of simple Enriques $2n$-folds.
What is this function on the moduli space?

Remark 2. When $n = 1$, τ_{Enr} is given by the Petersson norm of the Borcherds Φ-function.

To study this question, we compute the complex Hessian of τ_{Enr}.

Set up Y : simple Enriques $2n$-fold of index d

$\mathrm{Def}(Y)$: Kuranishi space of Y (smooth by Bogomolov-Tian-Todorov)

$f\colon (\mathfrak{Y}, Y) \to (\mathrm{Def}(Y), [Y])$: universal deformation of Y

$\Xi_{\mathfrak{Y}/\mathrm{Def}(Y)}$: relative pluricanonical form of weight d

Set up Y : simple Enriques $2n$-fold of index d
$\mathrm{Def}(Y)$: Kuranishi space of Y (smooth by Bogomolov-Tian-Todorov)
$f\colon (\mathfrak{Y}, Y) \to (\mathrm{Def}(Y), [Y])$: universal deformation of Y
$\Xi_{\mathfrak{Y}/\mathrm{Def}(Y)}$: relative pluricanonical form of weight d

Definition

Weil-Petersson form is the Kähler form on $\mathrm{Def}(Y)$ defined as

$$\omega_{\mathrm{WP}} := -dd^c \log \|\Xi_{\mathfrak{Y}/\mathrm{Def}(Y)}\|_{L^{2/d}}$$

Set up Y : simple Enriques $2n$-fold of index d

$\mathrm{Def}(Y)$: Kuranishi space of Y (smooth by Bogomolov-Tian-Todorov)

$f\colon (\mathfrak{Y}, Y) \to (\mathrm{Def}(Y), [Y])$: universal deformation of Y

$\Xi_{\mathfrak{Y}/\mathrm{Def}(Y)}$: relative pluricanonical form of weight d

Definition

Weil-Petersson form is the Kähler form on $\mathrm{Def}(Y)$ defined as

$$\omega_{\mathrm{WP}} := -dd^c \log \|\Xi_{\mathfrak{Y}/\mathrm{Def}(Y)}\|_{L^{2/d}}$$

Set

$$\nu_{n,d} := \begin{cases} \frac{1}{d} & \text{(Calabi-Yau type)}, \\ \frac{(n+1)(d-1)}{2nd} & \text{(hyperkähler type)}. \end{cases}$$

Set up Y : simple Enriques $2n$-fold of index d

$\mathrm{Def}(Y)$: Kuranishi space of Y (smooth by Bogomolov-Tian-Todorov)

$f\colon (\mathfrak{Y}, Y) \to (\mathrm{Def}(Y), [Y])$: universal deformation of Y

$\varXi_{\mathfrak{Y}/\mathrm{Def}(Y)}$: relative pluricanonical form of weight d

Definition

Weil-Petersson form is the Kähler form on $\mathrm{Def}(Y)$ defined as

$$\omega_{\mathrm{WP}} := -dd^c \log \|\varXi_{\mathfrak{Y}/\mathrm{Def}(Y)}\|_{L^{2/d}}$$

Set

$$\nu_{n,d} := \begin{cases} \frac{1}{d} & \text{(Calabi-Yau type)}, \\ \frac{(n+1)(d-1)}{2nd} & \text{(hyperkähler type)}. \end{cases}$$

Theorem

$\log \tau_{\mathrm{Enr}}$ *is a strictly plurisubharmonic function on* $\mathrm{Def}(Y)$ *such that*

$$dd^c \log \tau_{\mathrm{Enr}} = \nu_{n,d}\, \omega_{\mathrm{WP}}.$$

Remark This formula follows from the curvature formula for Quillen metrics due to Bismut-Gillet-Soulé (or its equivariant version due to Ma):

$$-dd^c \log \tau_{\mathfrak{Y}/S} + \sum_{q \geq 0} (-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S}) \right]^{(1,1)}$$

applied to the Kuranishi family $f \colon \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

Remark This formula follows from the curvature formula for Quillen metrics due to Bismut-Gillet-Soulé (or its equivariant version due to Ma):

$$-dd^c \log \tau_{\mathfrak{Y}/S} + \sum_{q \geq 0} (-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S}) \right]^{(1,1)}$$

applied to the Kuranishi family $f : \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

Corollary

Every family of simple Enriques 2n-folds without singular fibers over a compact complex manifold is isotrivial.

proof

Remark This formula follows from the curvature formula for Quillen metrics due to Bismut-Gillet-Soulé (or its equivariant version due to Ma):

$$-dd^c \log \tau_{\mathfrak{Y}/S} + \sum_{q\geq 0}(-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S})\right]^{(1,1)}$$

applied to the Kuranishi family $f\colon \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

Corollary

Every family of simple Enriques $2n$-folds without singular fibers over a compact complex manifold is isotrivial.

proof Let $f\colon \mathcal{Y} \to S$ be such a family. Then $\log \tau_{\mathrm{Enr}}$ for this family is a C^∞ plurisubharmonic (PSH) function on S.

Remark This formula follows from the curvature formula for Quillen metrics due to Bismut-Gillet-Soulé (or its equivariant version due to Ma):

$$-dd^c \log \tau_{\mathfrak{Y}/S} + \sum_{q\geq 0} (-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S}) \right]^{(1,1)}$$

applied to the Kuranishi family $f\colon \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

Corollary

Every family of simple Enriques 2n-folds without singular fibers over a compact complex manifold is isotrivial.

proof Let $f\colon \mathcal{Y} \to S$ be such a family. Then $\log \tau_{\mathrm{Enr}}$ for this family is a C^∞ plurisubharmonic (PSH) function on S. If $f\colon \mathcal{Y} \to S$ is not isotrivial, the differential of the classifying map $\mu\colon S \ni s \to \mathrm{Def}(Y_s)$ does not vanish at some point of S.

Remark This formula follows from the curvature formula for Quillen metrics due to Bismut-Gillet-Soulé (or its equivariant version due to Ma):

$$-dd^c \log \tau_{\mathfrak{Y}/S} + \sum_{q\geq 0}(-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S})\right]^{(1,1)}$$

applied to the Kuranishi family $f\colon \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

Corollary

Every family of simple Enriques 2n-folds without singular fibers over a compact complex manifold is isotrivial.

proof Let $f\colon \mathcal{Y} \to S$ be such a family. Then $\log \tau_{\mathrm{Enr}}$ for this family is a C^∞ plurisubharmonic (PSH) function on S. If $f\colon \mathcal{Y} \to S$ is not isotrivial, the differential of the classifying map $\mu\colon S \ni s \to \mathrm{Def}(Y_s)$ does not vanish at some point of S. Hence $\mu^*\omega_{\mathrm{WP}}$ does not vanish at some point of S, so that $\log \tau_{\mathrm{Enr}}$ is not a constant on S.

Remark This formula follows from the curvature formula for Quillen metrics due to Bismut-Gillet-Soulé (or its equivariant version due to Ma):

$$-dd^c \log \tau_{\mathfrak{Y}/S} + \sum_{q\geq 0}(-1)^q c_1(R^q f_* \mathcal{O}_{\mathfrak{Y}}, h_{L^2}) = \left[f_* \mathrm{Td}(T\mathfrak{Y}/S, h_{\mathfrak{Y}/S})\right]^{(1,1)}$$

applied to the Kuranishi family $f\colon \mathfrak{Y} \to S = \mathrm{Def}(Y)$.

Corollary

Every family of simple Enriques 2n-folds without singular fibers over a compact complex manifold is isotrivial.

proof Let $f\colon \mathcal{Y} \to S$ be such a family. Then $\log \tau_{\mathrm{Enr}}$ for this family is a C^∞ plurisubharmonic (PSH) function on S. If $f\colon \mathcal{Y} \to S$ is not isotrivial, the differential of the classifying map $\mu\colon S \ni s \to \mathrm{Def}(Y_s)$ does not vanish at some point of S. Hence $\mu^*\omega_{\mathrm{WP}}$ does not vanish at some point of S, so that $\log \tau_{\mathrm{Enr}}$ is not a constant on S. If S is compact, this contradicts the fact that a PSH function on a compact complex manifold is constant. □

Theorem

Let $f : \mathcal{Y} \to C$ be a family of $2n$-folds over a compact Riemann surface C, whose general members are simple Enriques. If $0 \in C$ is a point of the discriminant locus, then there exists $\alpha \in \mathbb{Q}$ such that

$$\log \tau_{\mathrm{Enr}}(Y_s) = \alpha \log |s|^2 + O\left(\log(-\log |s|)\right) \qquad (s \to 0).$$

Theorem

Let $f: \mathcal{Y} \to C$ be a family of $2n$-folds over a compact Riemann surface C, whose general members are simple Enriques. If $0 \in C$ is a point of the discriminant locus, then there exists $\alpha \in \mathbb{Q}$ such that

$$\log \tau_{\mathrm{Enr}}(Y_s) = \alpha \log |s|^2 + O\left(\log(-\log |s|)\right) \qquad (s \to 0).$$

Remark

- This theorem is obtained by applying the Bismut-Lebeau embedding theorem for Quillen metrics to the family of embedings $Y_s \hookrightarrow \mathcal{Y}$.
- When the degeneration $f: (\mathcal{Y}, Y_0) \to (S, 0)$ is semistable, α can explicitly be evaluated as the integral of certain characteristic classes associated to the degeneration over certain divisors.
- It is also possible to compute the value α by using the embedding formula for equivariant Quillen metrics due to Bismut.

Quasi-affinity of the moduli space

Fact (Viehweg)

- *There is a coarse moduli space of polarized simple Enriques $2n$-folds of index d with Hilbert polynomial h.*
- $\mathcal{M}_h$ *: component of the moduli space* $\Longrightarrow \mathcal{M}_h$ *quasi-projective.*
- λ *: direct image of the d-th power of the relative canonical bundle* $\Longrightarrow \lambda$ *: ample line bundle on* $\mathcal{M}_h$*.*

Quasi-affinity of the moduli space

Fact (Viehweg)

- *There is a coarse moduli space of polarized simple Enriques* $2n$*-folds of index* d *with Hilbert polynomial* h*.*
- $\mathcal{M}_h$ *: component of the moduli space* $\Longrightarrow$ $\mathcal{M}_h$ *quasi-projective.*
- λ *: direct image of the* d*-th power of the relative canonical bundle* $\Longrightarrow$ λ *: ample line bundle on* $\mathcal{M}_h$*.*

$\mathcal{M}_h^*$: normalization of $\overline{\Phi_{|\lambda^\nu|}(\mathcal{M}_h)}$, $\nu \gg 1$ (BB like compactification)
$\mathcal{D}_h^* := \mathcal{M}_h^* \setminus \mathcal{M}_h$ (discriminant locus).

$$\overline{\mathcal{M}}_h := \mathcal{M}_h^* \setminus Z_h, \qquad Z_h := \{(\mathrm{Sing}\,\mathcal{M}_h^*) \cap \mathcal{D}_h^*\} \cup \mathrm{Sing}\,\mathcal{D}_h^*.$$

$\overline{\mathcal{M}}_h$ is a complex orbifold such that Z_h has codimension ≥ 2 in $\mathcal{M}_h^*$.
($\overline{\mathcal{M}}_h$ is an analogue of modular variety.)

Theorem

There exist $\ell \in \mathbb{Z}_{>0}$ *and a meromorphic section* σ *of* $\lambda^{2\ell d\nu_{n,d}} \otimes \chi$ *on* $\overline{\mathcal{M}}_h$*, where* $\chi \in H^1(\overline{\mathcal{M}}_h, S^1)$ *is a flat holomorphic line bundle, such that*

$$\tau_{\mathrm{Enr}}^{-2\ell d} = \|\sigma\|^2, \qquad \mathrm{Supp}\,(\mathrm{div}\sigma) \subset \mathcal{D}_h^*$$

Theorem

There exist $\ell \in \mathbb{Z}_{>0}$ *and a meromorphic section* σ *of* $\lambda^{2\ell d \nu_{n,d}} \otimes \chi$ *on* $\overline{\mathcal{M}}_h$*, where* $\chi \in H^1(\overline{\mathcal{M}}_h, S^1)$ *is a flat holomorphic line bundle, such that*

$$\tau_{\mathrm{Enr}}^{-2\ell d} = \|\sigma\|^2, \qquad \mathrm{Supp}\,(\mathrm{div}\sigma) \subset \mathcal{D}_h^*$$

Theorem

If the restriction map $H^1(\mathcal{M}_h^*, \mathbb{R}) \to H^1(\overline{\mathcal{M}}_h, \mathbb{R})$ *is surjective, then* $\mathcal{M}_h$ *is quasi-affine*

Theorem

There exist $\ell \in \mathbb{Z}_{>0}$ and a meromorphic section σ of $\lambda^{2\ell d \nu_{n,d}} \otimes \chi$ on $\overline{\mathcal{M}}_h$, where $\chi \in H^1(\overline{\mathcal{M}}_h, S^1)$ is a flat holomorphic line bundle, such that

$$\tau_{\mathrm{Enr}}^{-2\ell d} = \|\sigma\|^2, \qquad \mathrm{Supp}\,(\mathrm{div}\sigma) \subset \mathcal{D}_h^*$$

Theorem

If the restriction map $H^1(\mathcal{M}_h^*, \mathbb{R}) \to H^1(\overline{\mathcal{M}}_h, \mathbb{R})$ is surjective, then $\mathcal{M}_h$ is quasi-affine

Remark When the type is hyperkähler, $\mathcal{M}_h$ is often quasi-affine without the assumption of surjectivity $H^1(\mathcal{M}_h^*, \mathbb{R}) \to H^1(\overline{\mathcal{M}}_h, \mathbb{R})$.

Theorem

There exist $\ell \in \mathbb{Z}_{>0}$ *and a meromorphic section* σ *of* $\lambda^{2\ell d \nu_{n,d}} \otimes \chi$ *on* $\overline{\mathcal{M}}_h$*, where* $\chi \in H^1(\overline{\mathcal{M}}_h, S^1)$ *is a flat holomorphic line bundle, such that*

$$\tau_{\mathrm{Enr}}^{-2\ell d} = \|\sigma\|^2, \qquad \mathrm{Supp}\,(\mathrm{div}\sigma) \subset \mathcal{D}_h^*$$

Theorem

If the restriction map $H^1(\mathcal{M}_h^*, \mathbb{R}) \to H^1(\overline{\mathcal{M}}_h, \mathbb{R})$ *is surjective, then* $\mathcal{M}_h$ *is quasi-affine*

Remark When the type is hyperkähler, $\mathcal{M}_h$ is often quasi-affine without the assumption of surjectivity $H^1(\mathcal{M}_h^*, \mathbb{R}) \to H^1(\overline{\mathcal{M}}_h, \mathbb{R})$.

question

Is $\mathcal{M}_h$ always quasi-affine?

Automorphy of τ_{Enr} : hyperkähler type case

Y : simple Enriques $2n$-fold of hyperkähler type with index 2
X : universal covering hyperkähler manifold
σ : holomorphic symplectic form on X

Automorphy of τ_{Enr} : hyperkähler type case

Y : simple Enriques $2n$-fold of hyperkähler type with index 2
X : universal covering hyperkähler manifold
σ : holomorphic symplectic form on X

Fact (Beauville, Bogomolov, Fujiki)

There is an integral non-degenerate symmetric bilinear form $(\cdot,\cdot)_X$ *on* $H^2(X,\mathbb{Z})$ *and a constant* $c_X \in \mathbb{Q}_{>0}$ *such that*

(1) $\int_X \lambda^{2n} = c_X \cdot (\lambda,\lambda)_X^n \quad (\forall\, \lambda \in H^2(X,\mathbb{Z}))$

(2) $(\sigma,\sigma)_X = 0, \quad (\sigma,\overline{\sigma})_X > 0$

(3) $\mathrm{sign}\,(\cdot,\cdot)_X = (3, b_2 - 3)$

$(H^2(X,\mathbb{Z}), (\cdot,\cdot)_X)$ *is called the Beauville-Bogomolov-Fujiki lattice*

Automorphy of τ_{Enr} : hyperkähler type case

Y : simple Enriques $2n$-fold of hyperkähler type with index 2
X : universal covering hyperkähler manifold
σ : holomorphic symplectic form on X

Fact (Beauville, Bogomolov, Fujiki)

There is an integral non-degenerate symmetric bilinear form $(\cdot,\cdot)_X$ *on* $H^2(X,\mathbb{Z})$ *and a constant* $c_X \in \mathbb{Q}_{>0}$ *such that*

(1) $\int_X \lambda^{2n} = c_X \cdot (\lambda,\lambda)_X^n \quad (\forall\, \lambda \in H^2(X,\mathbb{Z}))$

(2) $(\sigma,\sigma)_X = 0, \quad (\sigma,\overline{\sigma})_X > 0$

(3) $\mathrm{sign}\,(\cdot,\cdot)_X = (3, b_2 - 3)$

$(H^2(X,\mathbb{Z}), (\cdot,\cdot)_X)$ *is called the Beauville-Bogomolov-Fujiki lattice*

Assumption

$$\mathrm{Aut}_0(X) := \ker\left\{\mathrm{Aut}(X) \to \mathrm{Aut}\left(H^2(X,\mathbb{Z})\right)\right\} = \{1\}, \qquad b_2(X) \geq 5$$

Λ : lattice isometric to the BBF lattice of X.
An isometry $\alpha\colon H^2(X,\mathbb{Z}) \to \Lambda$ is called a marking.
$\iota\colon X \to X$: involution such that $X/\iota = Y$
$I := \alpha \circ \iota^* \circ \alpha^{-1}$: involution on Λ induced by ι
Set

$$M := \{l \in \Lambda;\ I(l) = l\}, \qquad N := \{l \in \Lambda;\ I(l) = -l\}.$$

Λ : lattice isometric to the BBF lattice of X.
An isometry $\alpha\colon H^2(X,\mathbb{Z}) \to \Lambda$ is called a marking.
$\iota\colon X \to X$: involution such that $X/\iota = Y$
$I := \alpha \circ \iota^* \circ \alpha^{-1}$: involution on Λ induced by ι
Set

$$M := \{l \in \Lambda;\ I(l) = l\}, \qquad N := \{l \in \Lambda;\ I(l) = -l\}.$$

Fact

- *M is Lorentzian.*
- $\mathrm{sign}(N) = (2, r(N) - 2)$.

Λ : lattice isometric to the BBF lattice of X.
An isometry $\alpha\colon H^2(X,\mathbb{Z}) \to \Lambda$ is called a marking.
$\iota\colon X \to X$: involution such that $X/\iota = Y$
$I := \alpha \circ \iota^* \circ \alpha^{-1}$: involution on Λ induced by ι
Set

$$M := \{l \in \Lambda;\ I(l) = l\}, \qquad N := \{l \in \Lambda;\ I(l) = -l\}.$$

Fact

- *M is Lorentzian.*
- $\mathrm{sign}(N) = (2, r(N) - 2)$.

Definition

Period domain for simple Enriques $2n$-folds deformation equivalent to Y is the domain of type IV defined by

$$\Omega_N := \{[\sigma] \in \mathbb{P}(N \otimes \mathbb{C});\ (\sigma,\sigma)_N = 0, \quad (\sigma,\overline{\sigma})_N > 0\}$$

Fact (Joumaah)

The Hausdorff reduction of the moduli space of (unpolarized) simple Enriques $2n$-folds deformation equivalent to Y is given by

$$(\Omega_N - \mathcal{D}_N^{\mathrm{MBM}})/\Gamma_{N,[\mathcal{K}]}$$

where $\mathcal{D}_N^{\mathrm{MBM}} = \bigcup_{d \in \Delta_N^{\mathrm{MBM}}} d^{\perp}$ is a divisor on Ω_N determined by the MBM classes in N, $\Gamma_{N,[\mathcal{K}]} \subset O(N)$ is a subgroup of finite index and $[\mathcal{K}]$ is a datum encoding the deformation equivalence class of (X, ι).

Fact (Joumaah)

The Hausdorff reduction of the moduli space of (unpolarized) simple Enriques $2n$-folds deformation equivalent to Y is given by

$$(\Omega_N - \mathcal{D}_N^{\mathrm{MBM}})/\Gamma_{N,[\mathcal{K}]}$$

where $\mathcal{D}_N^{\mathrm{MBM}} = \bigcup_{d \in \Delta_N^{\mathrm{MBM}}} d^\perp$ is a divisor on Ω_N determined by the MBM classes in N, $\Gamma_{N,[\mathcal{K}]} \subset O(N)$ is a subgroup of finite index and $[\mathcal{K}]$ is a datum encoding the deformation equivalence class of (X, ι).

Fact (Joumaah)

- *There is a chamber structure of the positive cone of M, whose chamber corresponds to the invariant Kähler cone of some (X, ι).*
- *There is an arithmetic subgroup $\Gamma_M \subset O(M)$ such that $[\mathcal{K}]$ is the Γ_M-orbit of a chamber $\mathcal{K}$.*

Theorem

There exist an integer $\nu \in \mathbb{Z}_{>0}$ *and a (possibly meromorphic) automorphic form* $\Phi_{N,[\mathcal{K}]}$ *on* Ω_N^+ *for* $\Gamma_{N,[\mathcal{K}]}$ *of weight* $\nu(n+1)/4$ *such that*

$$\tau_{\mathrm{Enr},[\mathcal{K}]}^{-\nu} = \left\|\Phi_{N,[\mathcal{K}]}\right\|^2, \qquad \mathrm{div}\left(\Phi_{N,[\mathcal{K}]}\right) \subset \mathcal{D}_N^{\mathrm{MBM}}$$

Theorem

There exist an integer $\nu \in \mathbb{Z}_{>0}$ *and a (possibly meromorphic) automorphic form* $\Phi_{N,[\mathcal{K}]}$ *on* Ω_N^+ *for* $\Gamma_{N,[\mathcal{K}]}$ *of weight* $\nu(n+1)/4$ *such that*

$$\tau_{\mathrm{Enr},[\mathcal{K}]}^{-\nu} = \left\|\Phi_{N,[\mathcal{K}]}\right\|^2, \qquad \mathrm{div}\left(\Phi_{N,[\mathcal{K}]}\right) \subset \mathcal{D}_N^{\mathrm{MBM}}$$

Conjecture (elliptic modularity)

- $\Gamma_{N,[\mathcal{K}]}$ contains $\widetilde{O}(N) := \ker\{O(N) \to O(N^\vee/N, q_N)\}$.
- $\Phi_{N,[\mathcal{K}]}$ is a Borcherds product: There is an elliptic modular form of type ρ_N and weight $(4 - r(N))/2$, whose Borcherds lift is $\Phi_{N,[\mathcal{K}]}$.

When

$$\operatorname{div} \Phi_{N,[\mathcal{K}]} = \sum_{d \in \Delta_N^{\mathrm{MBM}}} c(d)\, d^{\perp}, \qquad c(d) \in \mathbb{Z},$$

the slope of $\Phi_{N,[\mathcal{K}]}$ is defined as

$$\operatorname{slope}(\Phi_{N,[\mathcal{K}]}) := \max_{d \in \Delta_N^{\mathrm{MBM}}} \frac{c(d)}{\operatorname{wt}(\Phi_{N,[\mathcal{K}]})}$$

When

$$\operatorname{div}\Phi_{N,[\mathcal{K}]} = \sum_{d\in\Delta_N^{\mathrm{MBM}}} c(d)\, d^{\perp}, \qquad c(d)\in\mathbb{Z},$$

the slope of $\Phi_{N,[\mathcal{K}]}$ is defined as

$$\mathrm{slope}(\Phi_{N,[\mathcal{K}]}) := \max_{d\in\Delta_N^{\mathrm{MBM}}} \frac{c(d)}{\mathrm{wt}(\Phi_{N,[\mathcal{K}]})}$$

Conjecture (reflectivity)

- $\Phi_{N,[\mathcal{K}]}$ is holomorphic and reflective, i.e., $\mathrm{div}(\Phi_{N,[\mathcal{K}]})$ is contained in the ramification divisor of the projection $\Omega_N \to \Omega_N/\Gamma_{N,[\mathcal{K}]}$.
- There is a universal bound of $\mathrm{slope}(\Phi_{N,[\mathcal{K}]})$: $\exists$ an absolute constant $C>0$ such that for any deformation type of Enriques manifolds

$$0 < \mathrm{slope}(\Phi_{N,[\mathcal{K}]}) < C.$$

When

$$\operatorname{div} \Phi_{N,[\mathcal{K}]} = \sum_{d \in \Delta_N^{\mathrm{MBM}}} c(d)\, d^{\perp}, \qquad c(d) \in \mathbb{Z},$$

the slope of $\Phi_{N,[\mathcal{K}]}$ is defined as

$$\mathrm{slope}(\Phi_{N,[\mathcal{K}]}) := \max_{d \in \Delta_N^{\mathrm{MBM}}} \frac{c(d)}{\mathrm{wt}(\Phi_{N,[\mathcal{K}]})}$$

Conjecture (reflectivity)

- $\Phi_{N,[\mathcal{K}]}$ is holomorphic and reflective, i.e., $\mathrm{div}(\Phi_{N,[\mathcal{K}]})$ is contained in the ramification divisor of the projection $\Omega_N \to \Omega_N/\Gamma_{N,[\mathcal{K}]}$.
- There is a universal bound of $\mathrm{slope}(\Phi_{N,[\mathcal{K}]})$: $\exists$ an absolute constant $C > 0$ such that for any deformation type of Enriques manifolds

$$0 < \mathrm{slope}(\Phi_{N,[\mathcal{K}]}) < C.$$

Remark If this conjecture holds true, there are only finitely many possibilities of N, up to a scaling and isometry, by a result of S. Ma.

Enriques varieties of Boissière-Nieper-Weisskirchen-Sarti

$A = (A_1, \ldots, A_{m+1}), B = (B_1, \ldots, B_{m+1}) \in \mathrm{Sym}(m+1, \mathbb{C}) \otimes \mathbb{C}^{m+1}$
A_i, B_j : complex $(m+1) \times (m+1)$-symmetric matrices
$Q(x, A_i)$, $Q(y, B_j)$: quadratic forms associated with A_i, B_j.
Define a $(2, \ldots, 2)$-complete intersection

$$X_{(A,B)} := \left\{(x, y) \in \mathbb{P}^{2m+1};\ Q(x, A_i) + Q(y, B_i) = 0 \quad (1 \leq i \leq m+1)\right\}.$$

Enriques varieties of Boissière-Nieper-Weisskirchen-Sarti

$A = (A_1, \ldots, A_{m+1}), B = (B_1, \ldots, B_{m+1}) \in \mathrm{Sym}(m+1, \mathbb{C}) \otimes \mathbb{C}^{m+1}$
A_i, B_j : complex $(m+1) \times (m+1)$-symmetric matrices
$Q(x, A_i)$, $Q(y, B_j)$: quadratic forms associated with A_i, B_j.
Define a $(2, \ldots, 2)$-complete intersection

$$X_{(A,B)} := \left\{ (x, y) \in \mathbb{P}^{2m+1};\ Q(x, A_i) + Q(y, B_i) = 0 \quad (1 \leq i \leq m+1) \right\}.$$

Fact

A, B sufficiently general $\Longrightarrow X_{(A,B)}$ Calabi-Yau manifold of dimension m.

Enriques varieties of Boissière-Nieper-Weisskirchen-Sarti

$A = (A_1, \dots, A_{m+1}), B = (B_1, \dots, B_{m+1}) \in \mathrm{Sym}(m+1, \mathbb{C}) \otimes \mathbb{C}^{m+1}$
A_i, B_j : complex $(m+1) \times (m+1)$-symmetric matrices
$Q(x, A_i)$, $Q(y, B_j)$: quadratic forms associated with A_i, B_j.
Define a $(2, \dots, 2)$-complete intersection

$$X_{(A,B)} := \left\{(x, y) \in \mathbb{P}^{2m+1};\ Q(x, A_i) + Q(y, B_i) = 0 \quad (1 \leq i \leq m+1)\right\}.$$

Fact

A, B sufficiently general $\Longrightarrow X_{(A,B)}$ Calabi-Yau manifold of dimension m.

$\iota \colon \mathbb{P}^{2m+1} \to \mathbb{P}^{2m+1}$: involution defined by

$$\iota(x, y) := (x, -y)$$

Then ι acts on $X_{(A,B)}$. Define

$$Y_{(A,B)} := X_{(A,B)}/\iota$$

$R(A)$: resultant of the system of quadratics $Q(x, A_1), \ldots, Q(x, A_{m+1})$

$R(A)$: resultant of the system of quadratics $Q(x, A_1), \dots, Q(x, A_{m+1})$

Fact

- $X^{\iota}_{(A,B)} \neq \emptyset \iff R(A)R(B) = 0$
- *A, B sufficiently general, $R(A)R(B) \neq 0$, m even $\Longrightarrow Y_{(A,B)}$ simple Enriques m-fold of Calabi-Yau type.*

$R(A)$: resultant of the system of quadratics $Q(x, A_1), \dots, Q(x, A_{m+1})$

Fact

- $X^{\iota}_{(A,B)} \neq \emptyset \iff R(A)R(B) = 0$
- *A, B sufficiently general, $R(A)R(B) \neq 0$, m even $\implies Y_{(A,B)}$ simple Enriques m-fold of Calabi-Yau type.*

Theorem

For even m, there is a constant C_m depending only on m such that for $A, B \in \mathrm{Sym}(m+1, \mathbb{C}) \otimes \mathbb{C}^{m+1}$ sufficiently general with $R(A)R(B) \neq 0$,

$$\tau_{\mathrm{Enr}}\left(Y_{(A,B)}\right)^{-2^{m+1}} = C_m |R(A)R(B)| \left| \int_{X_{(A,B)}} \omega_{(A,B)} \wedge \overline{\omega}_{(A,B)} \right|^{2^m}$$

where $\omega_{(A,B)}$ is the canonical form on $X_{(A,B)}$ defined as the residue of

$$Q(x, A_1) + Q(y, B_1), \dots, Q(x, A_{m+1}) + Q(y, B_{m+1})$$

Enriques varieties parametrized by configuration space

g : *even* positive integer

For $N = (\mathbf{n}_1, \ldots, \mathbf{n}_{2g+2}) \in M_{g+1,2g+2}(\mathbb{C})$, $\mathbf{n}_i \in \mathbb{C}^{g+1}$ $(1 \leq i \leq 2g+2)$,

$$X_N := \{[x] \in \mathbb{P}^{2g+1};\ \sum_{i=1}^{2g+2} x_i^2 \mathbf{n}_i = \mathbf{0}\}$$

is a Calabi-Yau g-fold, when N is sufficiently general.

Enriques varieties parametrized by configuration space

g : *even* positive integer

For $N = (\mathbf{n}_1, \ldots, \mathbf{n}_{2g+2}) \in M_{g+1,2g+2}(\mathbb{C})$, $\mathbf{n}_i \in \mathbb{C}^{g+1}$ $(1 \leq i \leq 2g+2)$,

$$X_N := \{[x] \in \mathbb{P}^{2g+1};\ \sum_{i=1}^{2g+2} x_i^2 \mathbf{n}_i = \mathbf{0}\}$$

is a Calabi-Yau g-fold, when N is sufficiently general.

For $J = \{j_1 < \cdots < j_{g+1}\} \subset \{1, \ldots, 2g+2\}$, let $\langle J \rangle$ be the partition

$$\langle J \rangle := J \amalg J^c = \{1, \ldots, 2g+2\}.$$

Enriques varieties parametrized by configuration space

g : *even* positive integer

For $N = (\mathbf{n}_1, \dots, \mathbf{n}_{2g+2}) \in M_{g+1,2g+2}(\mathbb{C})$, $\mathbf{n}_i \in \mathbb{C}^{g+1}$ $(1 \leq i \leq 2g+2)$,

$$X_N := \{[x] \in \mathbb{P}^{2g+1};\ \sum_{i=1}^{2g+2} x_i^2 \mathbf{n}_i = \mathbf{0}\}$$

is a Calabi-Yau g-fold, when N is sufficiently general.

For $J = \{j_1 < \cdots < j_{g+1}\} \subset \{1, \dots, 2g+2\}$, let $\langle J \rangle$ be the partition

$$\langle J \rangle := J \amalg J^c = \{1, \dots, 2g+2\}.$$

For each partition $\langle J \rangle$, define an involution $\iota_{\langle J \rangle} : \mathbb{P}^{2g+1} \to \mathbb{P}^{2g+1}$ by

$$\iota_{\langle J \rangle}(x_J, x_{J^c}) := (x_J, -x_{J^c})$$

Then $\iota_{\langle J \rangle}$ acts on X_N. We set

$$Y_{N,\langle J \rangle} := X_N / \iota_{\langle J \rangle}.$$

Fact

- *$X_N \cong X_{N'}$ if N and N' lie in the same orbit of $\mathrm{GL}(\mathbb{C}^g) \times (\mathbb{C}^*)^{2g+2}$*
- $X_N^{\iota\langle J\rangle} = \emptyset \iff \Delta_{\langle J\rangle}(N) := \det(\mathbf{n}_{j_1}, \ldots, \mathbf{n}_{j_{g+1}}) \det(\mathbf{n}_{j_1^c}, \ldots, \mathbf{n}_{j_{g+1}^c}) \neq 0$
- *$Y_{N,\langle J\rangle}$ is a simple Enriques g-fold of Calabi-Yau type for all $\langle J\rangle \iff$ none of $(g+1) \times (g+1)$-minors of N vanishes*

Fact

- $X_N \cong X_{N'}$ *if* N *and* N' *lie in the same orbit of* $\mathrm{GL}(\mathbb{C}^g) \times (\mathbb{C}^*)^{2g+2}$
- $X_N^{\iota\langle J\rangle} = \emptyset \iff \Delta_{\langle J\rangle}(N) := \det(\mathbf{n}_{j_1}, \ldots, \mathbf{n}_{j_{g+1}}) \det(\mathbf{n}_{j_1^c}, \ldots, \mathbf{n}_{j_{g+1}^c}) \neq 0$
- $Y_{N,\langle J\rangle}$ *is a simple Enriques g-fold of Calabi-Yau type for all* $\langle J\rangle \iff$ *none of* $(g+1) \times (g+1)$*-minors of* N *vanishes*

Theorem

For all $N \in M^o_{g+1,2g+2}(\mathbb{C}) := M_{g+1,2g+2}(\mathbb{C}) \setminus \bigcup_{\langle J\rangle} \mathrm{div}(\Delta_{\langle J\rangle})$ *and* $\langle J\rangle$,

$$\tau_{\mathrm{Enr}}\left(Y_{N,\langle J\rangle}\right)^{-2^{g+1}} = C_g \left|\Delta_{\langle J\rangle}(N) \int_{X_N} \omega_N \wedge \overline{\omega}_N\right|^{2^g}.$$

Fact

- $X_N \cong X_{N'}$ *if* N *and* N' *lie in the same orbit of* $\mathrm{GL}(\mathbb{C}^g) \times (\mathbb{C}^*)^{2g+2}$
- $X_N^{\iota\langle J\rangle} = \emptyset \iff \Delta_{\langle J\rangle}(N) := \det(\mathbf{n}_{j_1}, \ldots, \mathbf{n}_{j_{g+1}}) \det(\mathbf{n}_{j_1^c}, \ldots, \mathbf{n}_{j_{g+1}^c}) \neq 0$
- $Y_{N,\langle J\rangle}$ *is a simple Enriques* g*-fold of Calabi-Yau type for all* $\langle J\rangle \iff$ *none of* $(g+1) \times (g+1)$*-minors of* N *vanishes*

Theorem

For all $N \in M^o_{g+1,2g+2}(\mathbb{C}) := M_{g+1,2g+2}(\mathbb{C}) \setminus \bigcup_{\langle J\rangle} \mathrm{div}(\Delta_{\langle J\rangle})$ *and* $\langle J\rangle$,

$$\tau_{\mathrm{Enr}}\left(Y_{N,\langle J\rangle}\right)^{-2^{g+1}} = C_g \left|\Delta_{\langle J\rangle}(N) \int_{X_N} \omega_N \wedge \overline{\omega}_N\right|^{2^g}.$$

Corollary

For all $N \in M^o_{g+1,2g+2}(\mathbb{C})$ *and partitions* $\langle J\rangle, \langle J'\rangle$,

$$\left\{\tau_{\mathrm{Enr}}\left(Y_{N,\langle J\rangle}\right) / \tau_{\mathrm{Enr}}\left(Y_{N,\langle J'\rangle}\right)\right\}^{-2} = \left|\Delta_{\langle J\rangle}(N)/\Delta_{\langle J'\rangle}(N)\right|.$$

$\lambda = (\lambda_1, \ldots, \lambda_{2g+2}) \in \mathbb{C}^{2g+2} \setminus \mathrm{div}(\Delta)$,
$\Delta(\lambda) := \prod_{i<j}(\lambda_j - \lambda_i)$ is the difference product of λ
Define a hyperelliptic curve C_λ by

$$C_\lambda := \{(x, y) \in \mathbb{C}^2;\ y^2 = (x - \lambda_1) \cdots (x - \lambda_{2g+2})\}.$$

Enriques varieties associated with hyperelliptic curves

$\lambda = (\lambda_1, \ldots, \lambda_{2g+2}) \in \mathbb{C}^{2g+2} \setminus \mathrm{div}(\Delta)$,
$\Delta(\lambda) := \prod_{i<j}(\lambda_j - \lambda_i)$ is the difference product of λ
Define a hyperelliptic curve C_λ by

$$C_\lambda := \{(x, y) \in \mathbb{C}^2;\ y^2 = (x - \lambda_1) \cdots (x - \lambda_{2g+2})\}.$$

Define a complex $(g+1) \times (2g+2)$-matrix of Vandermonde type

$$M(\lambda) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{2g+2} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{2g+2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^g & \lambda_2^g & \cdots & \lambda_{2g+2}^g \end{pmatrix}.$$

Enriques varieties associated with hyperelliptic curves

$\lambda = (\lambda_1, \ldots, \lambda_{2g+2}) \in \mathbb{C}^{2g+2} \setminus \mathrm{div}(\Delta)$,
$\Delta(\lambda) := \prod_{i<j}(\lambda_j - \lambda_i)$ is the difference product of λ
Define a hyperelliptic curve C_λ by

$$C_\lambda := \{(x, y) \in \mathbb{C}^2;\ y^2 = (x - \lambda_1)\cdots(x - \lambda_{2g+2})\}.$$

Define a complex $(g+1) \times (2g+2)$-matrix of Vandermonde type

$$M(\lambda) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{2g+2} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{2g+2}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^g & \lambda_2^g & \cdots & \lambda_{2g+2}^g \end{pmatrix}.$$

For $M(\lambda) = (M_1(\lambda), M_2(\lambda))$, $M_1(\lambda), M_2(\lambda) \in M_{g+1}(\mathbb{C})$, define

$$M(\lambda)^\vee := ({}^tM_1(\lambda)^{-1}, {}^tM_2(\lambda)^{-1})$$

Fact (Mumford)

There is a one-to-one correspondence:

$$\text{partitions } \{\langle J \rangle\} \Longleftrightarrow \text{non-vansihing even theta constants on } C_\lambda$$

Fact (Mumford)

There is a one-to-one correspondence:

$$\textit{partitions } \{\langle J\rangle\} \Longleftrightarrow \textit{non-vansihing even theta constants on } C_\lambda$$

Under this correspondence, write $\theta_{\langle J\rangle}(\Omega_\lambda)$ for the non-vanishing even theta constant on C_λ corresponding to the partition $\langle J\rangle$, where $\Omega_\lambda \in \mathfrak{S}_g$ is the period of C_λ w.r.t. a certain symplectic basis of $H_1(C_\lambda, \mathbb{Z})$.

Fact (Mumford)

There is a one-to-one correspondence:

$$\textit{partitions } \{\langle J\rangle\} \Longleftrightarrow \textit{non-vansihing even theta constants on } C_\lambda$$

Under this correspondence, write $\theta_{\langle J\rangle}(\Omega_\lambda)$ for the non-vanishing even theta constant on C_λ corresponding to the partition $\langle J\rangle$, where $\Omega_\lambda \in \mathfrak{S}_g$ is the period of C_λ w.r.t. a certain symplectic basis of $H_1(C_\lambda, \mathbb{Z})$.

Theorem

There is a constant $C(g)$ depending only on g such that for all $\lambda \in \mathbb{C}^{2g+2} \setminus \mathrm{div}(\Delta)$,

$$\tau_{\mathrm{Enr}}\left(Y_{M(\lambda)^\vee,\langle J\rangle}\right)^{-1} = C(g)\left\|\theta_{\langle J\rangle}(\Omega_\lambda)\right\|^2,$$

where $\|\theta_{\langle J\rangle}(\Omega_\lambda)\|$ is the Petersson norm of the theta constant $\theta_{\langle J\rangle}(\Omega_\lambda)$.

Question

Is $M(\lambda) = M(\lambda)^\vee$ in $\mathrm{GL}(\mathbb{C}^{g+1}) \backslash M^o_{g+1,2g+2}(\mathbb{C})/(\mathbb{C}^*)^{2g+2}$?

Question

Is $M(\lambda) = M(\lambda)^\vee$ in $\mathrm{GL}(\mathbb{C}^{g+1})\backslash M^o_{g+1,2g+2}(\mathbb{C})/(\mathbb{C}^*)^{2g+2}$?

Reamrk For Enriques Calabi-Yau threefolds $X_{(S,E)} = (\widetilde{S} \times E)/(\iota \times -1)$, we have the following relation: Up to a universal constant,

$$\tau_{\mathrm{BCOV}}\left(X_{(S,E)}\right) = \tau_{\mathrm{Enr}}(S)^{-8}\tau_{\mathrm{ell}}(E)^{-12}$$

Question

Is $M(\lambda) = M(\lambda)^\vee$ in $\mathrm{GL}(\mathbb{C}^{g+1})\backslash M^o_{g+1,2g+2}(\mathbb{C})/(\mathbb{C}^*)^{2g+2}$?

Reamrk For Enriques Calabi-Yau threefolds $X_{(S,E)} = (\widetilde{S} \times E)/(\iota \times -1)$, we have the following relation: Up to a universal constant,

$$\tau_{\mathrm{BCOV}}\left(X_{(S,E)}\right) = \tau_{\mathrm{Enr}}(S)^{-8}\tau_{\mathrm{ell}}(E)^{-12}$$

Question

What is the relation between the BCOV invariant of higher dimension (Eriksson-Freixas-Mourougane) and the invariant τ_{Enr}?

Question

Is $M(\lambda) = M(\lambda)^{\vee}$ in $\mathrm{GL}(\mathbb{C}^{g+1})\backslash M^{o}_{g+1,2g+2}(\mathbb{C})/(\mathbb{C}^{*})^{2g+2}$?

Reamrk For Enriques Calabi-Yau threefolds $X_{(S,E)} = (\widetilde{S} \times E)/(\iota \times -1)$, we have the following relation: Up to a universal constant,

$$\tau_{\mathrm{BCOV}}\left(X_{(S,E)}\right) = \tau_{\mathrm{Enr}}(S)^{-8}\tau_{\mathrm{ell}}(E)^{-12}$$

Question

What is the relation between the BCOV invariant of higher dimension (Eriksson-Freixas-Mourougane) and the invariant τ_{Enr}?

If τ_{Enr} is equivalent to τ_{BCOV} for the Enriques g-folds associated to hyperelliptic curves of even genus and if mirror symmetry at genus 1 holds true in higher dimension, then hyperelliptic even theta constants admits an infinite product expansion analogous to Borcherds products. ($g = 2$ case: Kawaguchi-Mukai-Y.)

Enriques manifolds of Oguiso-Schröer I

S : Enriques surface (compact Kähler, $K_S \not\cong \mathcal{O}_S$, $K_S^{\otimes 2} \cong \mathcal{O}_S$, $q(S) = 0$)
$\widetilde{S}$: universal covering $K3$ surface of S
$X := \mathrm{Hilb}^n(\widetilde{S})$: symplectic resolution of the symmetric product $\widetilde{S}^n/\mathfrak{S}_n$
$\epsilon\colon X \to \widetilde{S}^n/\mathfrak{S}_n$: resolution (Hilbert-Chow morphism)

Fact (Beauville)

X is a hyperkähler $2n$-fold such that

- $b_2(X) = 23$
- $(H^2(X,\mathbb{Z}), q_{\mathrm{BBF}}) \cong \Lambda_n := \mathbb{L}_{K3} \oplus \langle -2(n-1)\rangle$
- $\langle -2(n-1)\rangle$ *is generated by the* $1/2$ *of exceptional divisor* E *of* $\epsilon\colon X \to \widetilde{S}^n/\mathfrak{S}_n$
- $\mathrm{Aut}_0(X) = \{1\}$

where $\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$ *is the $K3$-lattice and* $\langle k\rangle$ *is the one-dimensional lattice* $(\mathbb{Z}, kx^2)$.

$\iota\colon \widetilde{S} \to \widetilde{S}$: non-trivial covering transformation
$\widetilde{\iota}\colon X \to X$: involution induced by ι
$H^2(X,\mathbb{Z})_\pm$: ± 1-eigenlattice of the $\widetilde{\iota}$-action on $H^2(X,\mathbb{Z})$
Set

$$M := \mathbb{U}(2) \oplus \mathbb{E}_8(2) \oplus \langle -2(n-1) \rangle, \qquad N := \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2).$$

Fact (Oguiso-Schröer)

- *n odd $\Longrightarrow Y := X/\widetilde{\iota}$ is a simple Enriques $2n$-fold of hyperkähler type*
- $H^2(X,\mathbb{Z})_+ \cong M, \qquad H^2(X,\mathbb{Z})_- \cong N$

Theorem

There is a constant C_n depending only on odd n such that

$$\tau_{\mathrm{Enr}}\left(\mathrm{Hilb}^n(\widetilde{S})/\widetilde{\iota}\right) = C_n \, \|\Phi(S)\|^{-\frac{n+1}{8}}$$

for every Enriques surface S, where $\|\Phi(S)\|$ is the Petersson norm of the Borcherds Φ-function evaluated at the period of S.

Enriques manifolds of Oguiso-Schröer II

S : Enriques surface, $\widetilde{S}$: universal covering $K3$ surface of S
$\mathbb{M}_{L3} := \mathbb{U}(-1) \oplus \mathbb{L}_{K3}$: the Mukai lattice such that $H(\widetilde{S}, \mathbb{Z}) \cong \mathbb{M}_{K3}$
$v = (v_0, v_1, v_2) \in H(\widetilde{S}, \mathbb{Z})$, H : ample line bundle on $\widetilde{S}$.
$M_H(v)$: moduli of H-stable torsion free coherent sheaves E on $\widetilde{S}$ with

$$\text{Mukai vector } v(E) := \mathrm{ch}(E)\sqrt{\mathrm{Td}(\widetilde{S})} = v$$

Fact (Mukai)

If v_1 is a primitive vector of $H(\widetilde{S}, \mathbb{Z})$ with $v^2 \geq 0$ and H is sufficiently general, then $M_H(v)$ is a hyperkähler manifold of dimension $v^2 + 2$.

Fact (Yoshioka)

$M_H(v)$ is deformation equivalent to $\mathrm{Hilb}^{v^2/2+1}(\widetilde{S})$. In particular, $\mathrm{Aut}_0(M_H(v)) = \{1\}$.

$\iota\colon \widetilde{S} \to \widetilde{S}$: non-trivial covering transformation
$\widetilde{\iota}$: involution on $M_H(v)$ induced by ι

Fact (Oguiso-Schröer)

If v is ι-invariant and $\chi(E)$ is odd, then $\widetilde{\iota}$ is free from fixed points and $M_H(v)/\widetilde{\iota}$ is a simple Enriques manifold with

$$H^2(M_H(v),\mathbb{Z})_- \cong H^2(S,\mathbb{Z})_- \cong \mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8(2).$$

Theorem

There is a constant $C_{[\mathcal{K}]}$ depending only on the deformation type $[\mathcal{K}]$ of $M_H(v)/\widetilde{\iota}$ such that for every Enriques surface S

$$\tau_{\mathrm{Enr}}\left(M_{(S,H)}(v)/\widetilde{\iota}\right) = C_{[\mathcal{K}]}\,\|\Phi(S)\|^{-\frac{v^2+4}{16}}$$

Enriques manifolds of Oguiso-Schröer III

S : Enriques surface

$Y := \mathrm{Hilb}^n(S)$: Hilbert scheme of 0-dim subschemes $Z \subset S$, $\mathrm{lg}(\mathcal{O}_Z) = n$

Fact (Oguiso-Schröer)

Y is a simple Enriques 2n-fold of Calabi-Yau type.

Theorem

There is a constant C'_n depending only on $n > 1$ such that

$$\tau_{\mathrm{Enr}}\left(\mathrm{Hilb}^n(S)\right) = C'_n \, \|\Phi(S)\|^{-\frac{n}{4}} .$$