Heights and periodic points for one-parameter families of Hénon maps

(Joint work with Liang-Chung Hsia)

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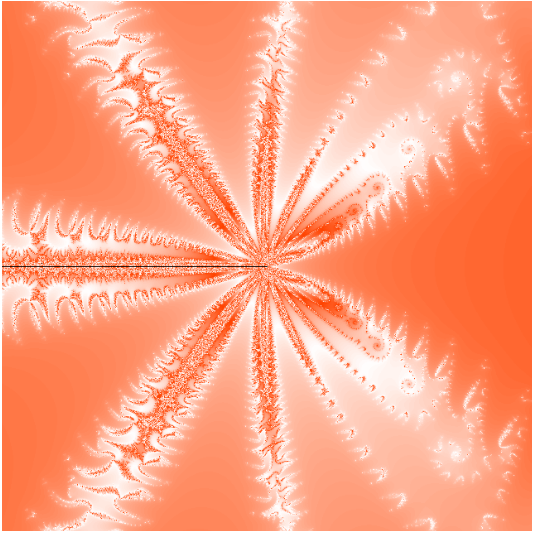
Plan of the talk

1. Background and motivation (surveyal):
   Variation of Néron–Tate heights for families of elliptic curves after Silverman, Tate, Masser–Zannier, DeMarco–Wang–Ye …
2. From elliptic curves to dynamical systems
3. Arithmetic properties of families of Hénon maps:
   Definition of a Hénon map, a height on the parameter space, the set of periodic parameter values, unlikely intersection …



Drawn by Qfract.

Part 1 Variation of Néron–Tate heights on elliptic curves

Variation of Néron–Tate heights

(For simplicity, we consider elliptic curves. For the general case of abelian varieties, see e.g. Call '89, Green '89, Holms–de Jong '15, '17)

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$\pi : \mathcal{E} \to B$ is an elliptic surface defined over a number field K,
B is a smooth projective curve
$E_t := \pi^{-1}(t)$ is a smooth elliptic curve
except for finitely many $t \in B(\overline{K})$.

Define $B^\circ \subseteq B$ to be the maximal Zariski open subset over which π is smooth.

Assume that π has a section $O : B \to \mathcal{E}$, which we regard as zero section.

Variation of Néron–Tate heights (continued)

$\mathcal{E} \to B$: an elliptic surface over a number field K
with zero section $O : B \to \mathcal{E}$

K is equipped with absolute values satisfying the product formula.
So is the function field $K(B)$ of B

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$\rightsquigarrow$ We have Néron–Tate heights (η: generic point of B)

- $\widehat{h}_{E_t} : E_t(\overline{K}) \to \mathbb{R}_{\geq 0}$ on each fiber E_t for $t \in B^\circ(\overline{K})$
- $\widehat{h}_{\mathcal{E}_\eta} : \mathcal{E}_\eta(\overline{K(B)}) \to \mathbb{R}_{\geq 0}$ on the generic fiber $\mathcal{E}_\eta$

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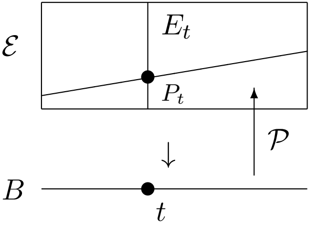
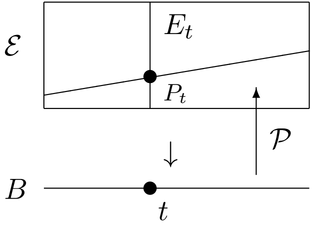
Let $\mathcal{P} : B \to \mathcal{E}$ be a section.
Set $P_t := \mathcal{P}(t)$ for $t \in B^\circ(\overline{K})$
How $\widehat{h}_{E_t}(P_t)$ and $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)$ are related?



Variation of Néron–Tate heights (continued)

$\mathcal{E} \to B$: an elliptic surface
over a number field K



Theorem (Silverman '83, Tate '83)

Let $\mathcal{P} : B \to \mathcal{E}$ be a section with $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \neq 0$.
Let h_B be a height on $B(\overline{K})$ associated to a degree 1 divisor. Then

$$\widehat{h}_{E_t}(P_t) = \widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)\, h_B(t) + O(\sqrt{h_B(t)}) \qquad \text{for any } t \in B^\circ(\overline{K}).$$

The error term $O(\sqrt{h_B(t)})$ is replaced by $O(1)$ if $B = \mathbb{P}^1$.

Variation of Néron–Tate heights (continued)

Assume $B = \mathbb{P}^1$, for simplicity of explanation.

$\mathcal{E} \to \mathbb{P}^1$: an elliptic surface over a number field K with zero section

$h_{\mathrm{std}} : \mathbb{P}^1(\overline{K}) \to \mathbb{R}_{\geq 0}$ standard logarithmic Weil height

$\mathcal{P} : \mathbb{P}^1 \to \mathcal{E}$ a section with $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \neq 0$

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Results of Silverman and Tate assert that

$$\widehat{h}_{E_t}(P_t) = \widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)\, h_{\mathrm{std}}(t) + O(1) \quad \text{for any } t \in (\mathbb{P}^1)^\circ(\overline{K}).$$

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We put $h_{\mathcal{P}} := \widehat{h}_{E_t}(P_t)/\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)$ for $t \in (\mathbb{P}^1)^\circ(\overline{K})$.

Then

$$h_{\mathcal{P}} = h_{\mathrm{std}} + O(1) \quad \text{on } (\mathbb{P}^1)^\circ(\overline{K}).$$

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Question What if $t \notin (\mathbb{P}^1)^\circ(\overline{K})$? (That is, what if E_t is singular?)

Variation of Néron–Tate heights (continued)

Theorem (DeMarco–Mavraki '17+ based on Silverman '92, '94)

Let $\mathcal{E} \to \mathbb{P}^1$ be an elliptic surface over a number field K with zero section. Let $\mathcal{P} : \mathbb{P}^1 \to \mathcal{E}$ be a section with $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \neq 0$. Set

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Then $h_{\mathcal{P}}$ is the restriction of a semipositive adelically metrized line bundle $\overline{\mathcal{L}_{\mathcal{P}}} = (\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_v\})$ on $\mathbb{P}^1$ (in the sense of Zhang).

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- Theorem says that $h_{\mathcal{P}}$ extends "nicely" to $t \notin (\mathbb{P}^1)^\circ(\overline{K})$ (That is, for t with singular E_t).

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- Theorem says that $h_{\mathcal{P}}$ extends "nicely" to $t \notin (\mathbb{P}^1)^\circ(\overline{K})$ (That is, for t with singular E_t).
- The base curve need not be $\mathbb{P}^1$. DeMarco–Mavraki showed that for any elliptic surface $\mathcal{E} \to B$ (B a smooth projective curve), $h_{\mathcal{P}}$ is the restriction of a semipositive adelically metrized line bundle on B.

Application to unlikely intersection of Masser–Zannier

Let $\mathcal{E} = \{y^2 z = x(x-z)(x-tz)\}$

Legendre family of elliptic curves over $t \in \mathbb{P}^1$.

(At $t = 0, 1, \infty$, E_t is singular)

For $a \in \overline{\mathbb{Q}} \setminus \{0,1\}$,

consider a section $\mathcal{P}_a : \mathbb{P}^1 \to \mathcal{E}$,

$$t \mapsto \Big((a : \sqrt{a(a-1)(a-t)} : 1), t \Big).$$

$$\Sigma(\mathcal{P}_a) := \{t \in \overline{\mathbb{Q}} \mid \mathcal{P}_a(t) \text{ is torsion on } E_t\}$$
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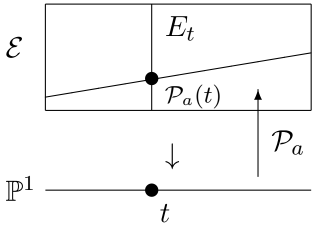

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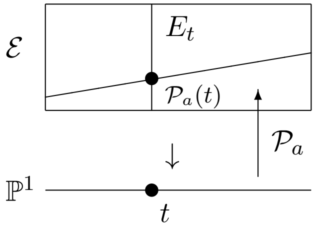

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Note: $\Sigma(\mathcal{P}_a)$ is an infinite set (Masser–Zannier).

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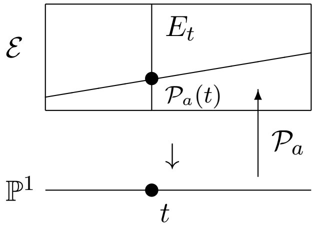

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Theorem (Masser–Zannier '08, '10, '12)

Let $a, b \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. If there are infinitely many parameter values $t \in \overline{\mathbb{Q}}$ such that both $\mathcal{P}_a(t)$ and $\mathcal{P}_b(t)$ are torsion points on E_t (i.e., if $\Sigma(\mathcal{P}_a) \cap \Sigma(\mathcal{P}_b)$ is an infinite set), then $a = b$.

Application to unlikely intersection of Masser–Zannier (continued)

DeMarco–Wang–Ye '14 give an alternate proof of Masser–Zannier's theorem.

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- $\Sigma(\mathcal{P}_a) := \{t \mid \mathcal{P}_a(t) \text{ torsion on } E_t\} = \{t \mid h_{\mathcal{P}_a}(t) = 0\}$.

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- $\Sigma(\mathcal{P}_a) := \{t \mid \mathcal{P}_a(t) \text{ torsion on } E_t\} = \{t \mid h_{\mathcal{P}_a}(t) = 0\}$.
- Since $h_{\mathcal{P}_a} = h_{\overline{\mathcal{L}_{\mathcal{P}_a}}}$ with semipositive adelically metrized line bundle $\overline{\mathcal{L}_{\mathcal{P}_a}}$, one can use the equidistribution theorem (such as Yuan. In [DeMarco–Wang–Ye], they use Baker–Rumely '06.).

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- If $|\Sigma(\mathcal{P}_a) \cap \Sigma(\mathcal{P}_b)| = \infty$, then equidistribution theorem implies that $c_1(\overline{\mathcal{L}_{\mathcal{P}_a}})_v = c_1(\overline{\mathcal{L}_{\mathcal{P}_b}})_v$ for any $v \in M_K$.

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- Then $h_{\overline{\mathcal{L}_{\mathcal{P}_a}}} = h_{\overline{\mathcal{L}_{\mathcal{P}_b}}}$, $\Sigma(\mathcal{P}_a) = \Sigma(\mathcal{P}_b)$, and (with more arguments) $a = b$.

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Part 2 From elliptic curves to dynamical systems

Canonical heights

E is an elliptic curve over a number field K

$L = \mathcal{O}_E([0])$, h_L is any height function associated to L

$[2]\colon E \to E$ twice multiplication map. Note that $[2]^*(L) \cong L^{\otimes 4}$.

Néron–Tate height $\widehat{h}_E : E(\overline{K}) \to \mathbb{R}_{\geq 0}$ is defined by

$$\widehat{h}_E(P) = \lim_{n\to\infty} \frac{1}{4^n} h_L([2]^n P)$$

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In place of $(E, [2], \mathcal{O}_E([0]))$, this construction of a height is generalized to the case (X, f, L)

(continued ...)

Canonical heights (continued)

X is a projective variety over a number field K

L is an ample line bundle over X

h_L is any height function associated to L

$f\colon X \to X$ a morphism

Assume that $f^*(L) \cong L^{\otimes d}$ for some $d > 1$

Such a triple (X, f, L) is called a polarized dynamical system

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Canonical height $\widehat{h}_f : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ is defined by

$$\widehat{h}_f(P) = \lim_{n\to\infty} \frac{1}{d^n} h_L(f^n P)$$

(Call–Silverman '93, Zhang '95)

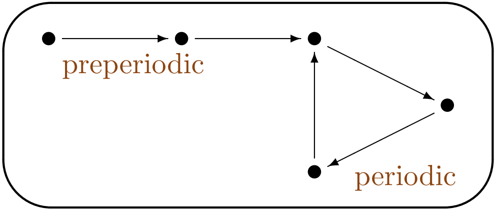
Néron–Tate height is when $(X, f, L) = (E, [2], \mathcal{O}_E([0]))$.

Torsion points, preperiodic points

$f : X \to X$ a morphism over a field K

A point $P \in X(\overline{K})$ is periodic if $f^n(P) = P$ for some $n \geq 1$

$P \in X(\overline{K})$ is preperiodic if $f^m(P)$ is periodic for some $m \geq 1$



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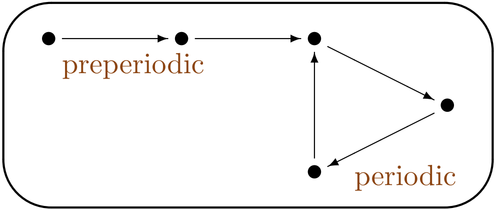

For an elliptic curve E, it is easy to see that a point $P \in E(\overline{K})$ is torsion if and only if P is preperiodic under [2].

Torsion points, preperiodic points (continued)

For an elliptic curve E over a number field K

$$\begin{aligned}\{\text{torsion point}\} &= \{\text{preperiodic point under } [2]\} \\ &= \{P \in E(\overline{K}) \mid \widehat{h}_E(P) = 0\}\end{aligned}$$

$$\Downarrow$$

Torsion points, preperiodic points (continued)

For an elliptic curve E over a number field K

$$
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\{\text{torsion point}\} &= \{\text{preperiodic point under } [2]\} \\
&= \{P \in E(\overline{K}) \mid \widehat{h}_E(P) = 0\}
\end{aligned}
$$

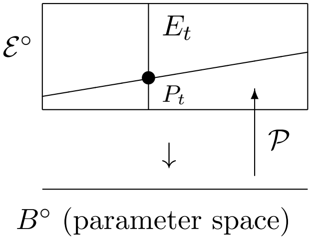
$$\Downarrow$$

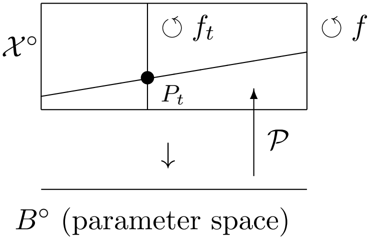
In a polarized dynamical system (X, f, L), in place of torsion points, we consider

$$\{\text{preperiodic point under } f\} = \{P \in X(\overline{K}) \mid \widehat{h}_f(P) = 0\}$$

(The equality follows from Northcott's finiteness theorem.)

Families (continued)





elliptic surface	polarized dynamical system
torsion point	preperiodic point
$\Sigma(\mathcal{P}) = \{t \mid P_t$ is torsion on $E_t\}$	$\Sigma(\mathcal{P}) = \{t \mid P_t$ is preperiodic under $f_t\}$
Néron–Tate height	canonical height

Families (continued)

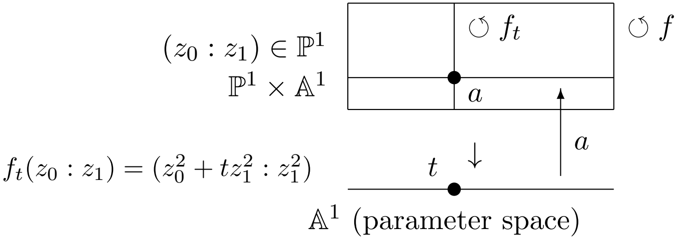
Baker–DeMarco obtained the the first result in a dynamical setting.

$$\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, \quad ((z_0 : z_1), t) \mapsto t$$

$$f : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{A}^1, \quad ((z_0 : z_1), t) \mapsto ((z_0^2 + t z_1^2 : z_1^2), t)$$

$$\text{a constant section } a : \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{A}^1, \quad t \mapsto ((a : 1), t)$$

$$\Sigma(a) := \{t \in \mathbb{A}^1 \mid (a : 1) \text{ is preperiodic under } f_t\}$$



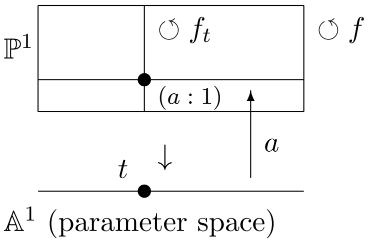
Families (continued)

$f_t(z_0 : z_1) = (z_0^2 + tz_1^2 : z_1^2)$

$\Sigma(a) = \{t \in \mathbb{A}^1(\mathbb{C})$

$\mid (a : 1)$ is preperiodic under $f_t\}$

Note: $\Sigma(a)$ is an infinite set.



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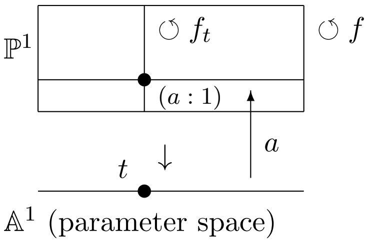

Theorem (Baker–DeMarco '11)

Let $f_t(z) = z^2 + t$. Let $a, b \in \mathbb{C}$. Suppose that there exist infinitely many $t \in \mathbb{C}$ such that a and b are both preperiodic under f_t. Then $a^2 = b^2$.

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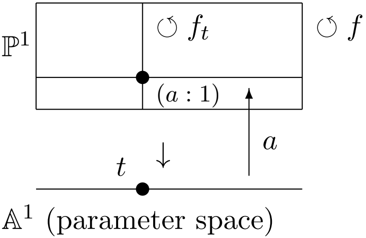

Theorem (Baker–DeMarco '11)

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- This answered a question of Zannier. The assumption says that $\#(\Sigma(a) \cap \Sigma(b)) = \infty$. Using equidistribution theorem, they show $\Sigma(a) = \Sigma(b)$. Properties of the Böttchcher coordinate then imply $a^2 = b^2$.

More comments on Baker–DeMarco's theorem

- Further generalizations (in relation to the dynamical Pink–Zilber conjecture) have been obtained by Baker–DeMarco, Ghioca–Hsia–Tucker, Favre–Gauthier, DeMarco–Wang–Ye ...
- Families of rational maps of $\mathbb{P}^1$ have been mostly studied. Our talk is about families of higher-dimensional maps.

Plan of the talk

1. Background and motivation (surveyal):
   Variation of Néron–Tate heights for families of elliptic curves after Silverman, Tate, Masser–Zannier, DeMarco–Wang–Ye . . .
2. From elliptic curves to dynamical systems
3. Arithmetic properties of families of Hénon maps:
   Definition of a Hénon map, a height on the parameter space, the set of periodic parameter values, unlikely intersection . . .

Part 3 Arithmetic properties of families of Hénon maps

Hénon maps

$\mathbb{A}^2$: affine plane

A Hénon map over a field K is an automorphism of the form

$$H : \mathbb{A}^2 \to \mathbb{A}^2, \quad H(x, y) = (\delta y + f(x), x)$$

for some $\delta \in K \setminus \{0\}$ and $f(x) \in K[x]$ with $d := \deg(f) \geq 2$.

The inverse is given by

$$H^{-1} : \mathbb{A}^2 \to \mathbb{A}^2, \quad H^{-1}(x, y) = \left(y, \frac{1}{\delta}(x - f(y)) \right)$$

Note: H extends to a birational map $H : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, but not an isomorphism.

Hénon maps (continued)

Some remarks on Hénon maps

- Hénon '76 showed that a Hénon map has a strange attractor.

Hénon maps (continued)

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- If δ in $H(x, y) = (\delta y + f(x), x)$ is very small (nearly 0), then the map looks like

$$(x, y) \mapsto (f(x), x) \mapsto (f^2(x), f(x)) \mapsto \dots$$

  So, Hénon maps are more complicated than one-variable polynomial maps.

Hénon maps (continued)

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  So, Hénon maps are more complicated than one-variable polynomial maps.
- Friedland–Milnor '89 showed that any automorphism $F : \mathbb{A}^2 \to \mathbb{A}^2$ over $\mathbb{C}$ is, up to conjugacy of $\mathrm{Aut}(\mathbb{C}^2)$, either triangularizable or the composition of Hénon maps. (Dynamically, triangularizable maps are not interesting.) So, the class of Hénon maps consists of a fundamental class of plane automorphisms.

Hénon maps (continued)

- Hénon maps over $\mathbb{C}$ are deeply studied in Bedford–Smilie ’91, Fornæss–Sibony ’92, Hubbard –Oberste-Vorth ’94 among from the vast literature.

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- Arithmetic properties of Hénon maps are also studied. To my best knowledge, they are first studied by Silverman '94.
- For a Hénon map $H(x, y) = (y, \delta x + f(x))$, up to conjugacy of $\mathrm{Aut}(\overline{K})$, we may assume that $f(x)$ is monic.

Canonical heights for Hénon maps

Hénon maps are not polarized dynamical systems,

but one can define canonical heights for Hénon maps

- Ω: an algebraically closed field complete with respect to an absolute value $|\cdot|$
- $\|(a_1, \ldots, a_n)\| := \max_i\{|a_i|\}$
- $\log^+(r) := \log\max\{r, 1\}$ for $r \in \mathbb{R}$
- $H\colon \mathbb{A}^2 \to \mathbb{A}^2$ a Hénon map over Ω, $P \in \mathbb{A}^2(\Omega)$

Definition (Green functions on $\mathbb{A}^2(\Omega)$)

$$G_H^+(P) := \lim_{n\to+\infty} \frac{1}{d^n} \log^+ \|H^n(P)\|, \; G_H^-(P) := \lim_{n\to+\infty} \frac{1}{d^n} \log^+ \|H^{-n}(P)\|$$

$$G_H(P) := \max\{G_H^+(P), G_H^-(P)\}$$

Canonical heights for Hénon maps (continued)

$H\colon \mathbb{A}^2 \to \mathbb{A}^2$ a Hénon map over a number field K

For each place $v \in M_K$ with absolute value $|\cdot|_v$,

K_v: completion of K with respect to $|\cdot|_v$

$\mathbb{K}_v$: completion of an algebraic closure of K_v

We have the v-adic Green function

$$G_{H,v} := \max\{G^+_{H,v}, G^-_{H,v}\} : \mathbb{A}^2(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}$$

Definition (canonical height)

$$\widetilde{h}_H\colon \mathbb{A}^2(\overline{K}) \to \mathbb{R}_{\geq 0}, \quad \widetilde{h}_H(P) := \sum_{v \in M_K} n_v G_{H,v}(P)$$

Here n_v is the usual normalizing constant.

For example, if $p \mid v$, then $n_v = [K_v : \mathbb{Q}_p]/[K : \mathbb{Q}]$.

Canonical heights for Hénon maps (continued)

$H : \mathbb{A}^2 \to \mathbb{A}^2$: a Hénon map of degree $d \geq 2$ over a number field K.

Canonical heights for Hénon maps (continued)

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- $G_{H,v} \colon \mathbb{A}^2(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}$ (v-adic Green function)
  $G_{H,v}(P) := \max\left\{\lim \frac{1}{d^n} \log^+ \|H^n(P)\|, \lim \frac{1}{d^n} \log^+ \|H^{-n}(P)\|\right\}$
- When $\mathbb{K}_v = \mathbb{C}$, the Green function is extremely useful in Bedford–Smilie '91, Farnæss–Sibony '92, Hubbard–Oberste-Vorth '94.
- $\widetilde{h}_H \colon \mathbb{A}^2(\overline{K}) \to \mathbb{R}_{\geq 0}$ (canonical height), $\quad \widetilde{h}_H(P) := \sum_{v \in M_K} n_v G_{H,v}$

Canonical heights for Hénon maps (continued)

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Theorem (K– '06, '13)

1. *The limits defining $G_{H,v}$ exist for all $v \in M_K$.*
2. $\widetilde{h}_H = h_{\mathrm{std}} + O(1)$ *on* $\mathbb{A}^2(\overline{K})$
3. $\{$*periodic point under* $H\} = \{P \in \mathbb{A}^2(\overline{K}) \mid \widetilde{h}_H(P) = 0\}$

Note: Hénon maps are automorphisms, so preperiodic = periodic.

Families (our setting)

K a number field, $\delta \in K \setminus \{0\}$,

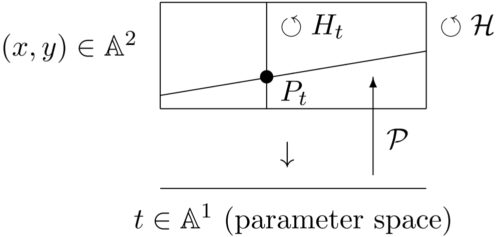
$f_t(x) \in K[t,x]$ monic in x, degree $d \geq 2$ in x

$$\mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^1, \quad ((x,y),t) \mapsto t$$

$$\mathcal{H} : \mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1, \quad ((x,y),t) \mapsto ((\delta y + f_t(x), x), t)$$

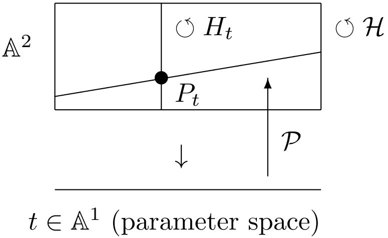
a section $\mathcal{P} \colon \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1, \quad t \mapsto ((a(t), b(t)), t)$

$$\Sigma(\mathcal{P}) := \{t \in \mathbb{A}^1(\overline{K}) \mid P_t = (a(t), b(t)) \text{ is periodic under } H_t\}$$



Families (continued)

K: a number field



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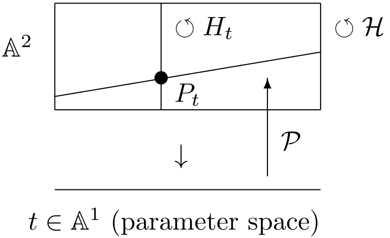


$\rightsquigarrow$ We have canonical heights (η: generic point of $\mathbb{A}^1$)

- $\widetilde{h}_{H_t} : \mathbb{A}^2(\overline{K}) \to \mathbb{R}_{\geq 0}$ for each $t \in \mathbb{A}^1(\overline{K})$
- $\widetilde{h}_{\mathcal{H}_\eta} : \mathbb{A}^2(\overline{K})(\overline{K(t)}) \to \mathbb{R}_{\geq 0}$ on the generic fiber $\mathbb{A}^2_\eta$

Let $\mathcal{P} : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1$ be a section with $\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$. Set

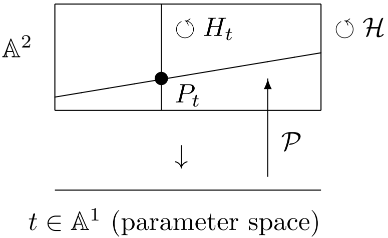
$$h_{\mathcal{P}}(t) := \widetilde{h}_{H_t}(P_t)/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta).$$

Question 1 Is $h_{\mathcal{P}} : \mathbb{A}^1(\overline{K}) \to \mathbb{R}$ a “nice” height function?

Families (continued)

K: a number field

$\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$



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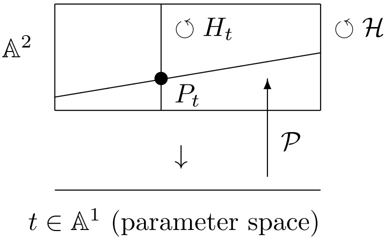


Theorem (Hsia–K)

$h_{\mathcal{P}}$ is the restriction of the height function associated to a semipositive adelically metrized line bundle $(\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_v\})$ on $\mathbb{P}^1$ (in the sense of Zhang).

Remark

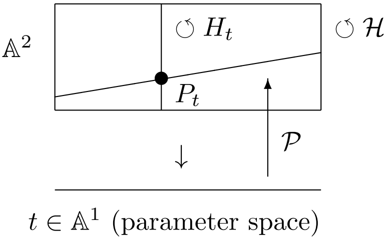
Ingram '14 showed a Silverman–Tate type estimate $\widetilde{h}_{H_t} = \widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) h_{\mathrm{std}} + O(1)$ on $\mathbb{A}^1(\overline{K})$. (His result is more general, and the base curve need not be $\mathbb{P}^1$.)

Families (continued)

K: a number field

$\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$

$v \in M_K$



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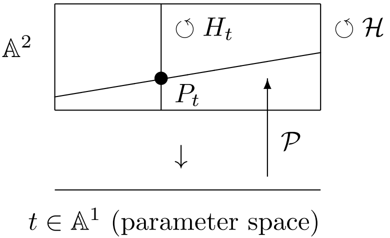

$$G_{\mathcal{P},v} \colon \mathbb{A}^1(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}, \qquad G_{\mathcal{P},v}(t) := G_{H_t,v}(P_t)$$

$$\mathcal{K}_{\mathcal{P},v} := \{t \in \mathbb{A}^1(\mathbb{K}_v) \mid \{H^n(P_t)\}_{n\in\mathbb{Z}} \text{ is bounded}\}$$

$$\mathcal{W}_{\mathcal{P},v} := \{t \in \mathbb{A}^1(\mathbb{K}_v) \mid \lim_{n\to+\infty} \|(H^n(P_t), H^{-n}(P_t))\| = +\infty\}$$

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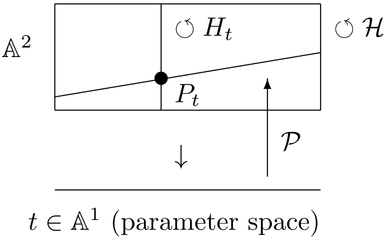

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Proposition (Hsia–K)

1. $h_{\mathcal{P}} = c\sum_{v\in M_K} n_v G_{\mathcal{P},v}$ with $c := \widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \in \mathbb{Q}_{>0}$.
2. $\mathcal{K}_{\mathcal{P},v} = \{t \in \mathbb{A}^1(\mathbb{K}_v) \mid G_{\mathcal{P},v}(t) = 0\}$
3. $\mathbb{A}^1(\mathbb{K}_v) = \mathcal{K}_{\mathcal{P},v} \amalg \mathcal{W}_{\mathcal{P},v}$

Example

$\mathcal{H} = (H_t)_t \colon \mathbb{A}^2 \to \mathbb{A}^2, \quad (x, y) \mapsto (y + x^2 + t, x)$

(family of quadratic Hénon maps parametrized by t)

$\mathcal{P}_{(0,0)} : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1$, $t \mapsto ((0,0), t)$ a constant family of initial points

$\mathcal{K}_{(0,0),\mathbb{C}} := \{t \in \mathbb{C} \mid \{H_t^n((0,0))\}_{n\in\mathbb{Z}} \text{ is bounded}\}$

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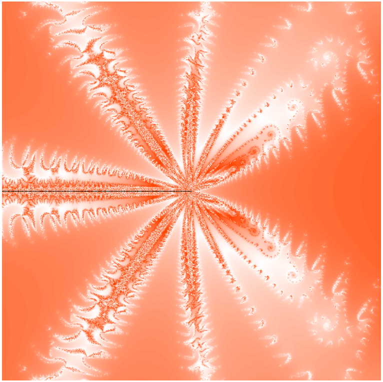
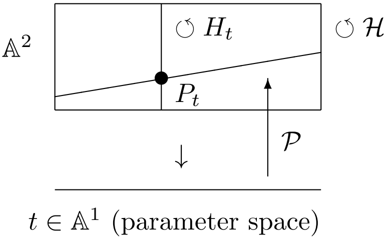


Figure: $|\mathrm{Re}(t)| \leq 0.1$, $|\mathrm{Im}(t)| \leq 0.1$

The set $\Sigma(\mathcal{P})$ of periodic parameter values

K: a number field

$\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$



$$\Sigma(\mathcal{P}) := \{t \in \mathbb{A}^1(\overline{K}) \mid P_t \text{ is periodic under } H_t\}$$

Here, we have a phenomenon that was not observed in families of one-dimensional dynamics.

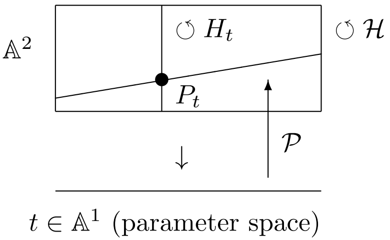
$\Sigma(\mathcal{P})$ may not be an infinite set.

Result on infiniteness of $\Sigma(\mathcal{P})$

K a field (of any characteristic)

$\mathcal{H}(x,y) = (\delta y + f_t(x), x)$

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
$\quad \mid P_t \text{ is periodic under } H_t\}$



Inspired by Dujardin–Favre's result on the dynamical Mordell–Lang conjecture for plane automorphisms, we consider a reversible Hénon map. Thus we assume that $\delta = \pm 1$ and that, if $\delta = 1$, then $f_t(x)$ is an even polynomial in x. Then, via the involution $\iota\colon (x,y) \mapsto (-\delta y, -\delta x)$, we have

$$\iota \circ \mathcal{H} \circ \iota = \mathcal{H}^{-1}.$$

Further ι has the fixed curve $C = \{x + \delta y = 0\}$ in $\mathbb{A}^2$.

Result on infiniteness of $\Sigma(\mathcal{P})$ (continued)

K a field (of any characteristic)

$\mathcal{H}(x,y) = (\delta y + f_t(x), x)$

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
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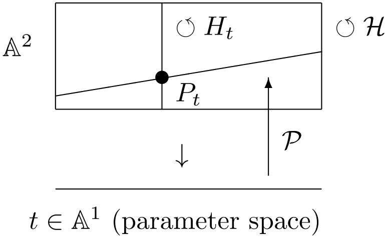


Theorem (Hsia–K)

We assume that $\delta = \pm 1$ and that, if $\delta = 1$, then $f_t(x)$ is an even polynomial in x. If the family of initial points $\mathcal{P} = (a(t), b(t))$ lie on the fixed curve of the involution $\iota\colon (x,y) \mapsto (-\delta y, -\delta x)$, i.e., $a(t) + \delta b(t) = 0$, then $\Sigma(\mathcal{P})$ is an infinite set.

Example

Let $\mathcal{H}(x,y) = (y + x^2 + t, x)$. Then, for any $a \in K$, $|\Sigma((a, -a))| = \infty$.

Result on finiteness/emptyness of $\Sigma(\mathcal{P})$

As a complimentary result, we point out that $\Sigma(\mathcal{P})$ can be the empty set.

Proposition (Hsia–K)

We consider the family of quadratic Hénon maps over $\mathbb{C}$

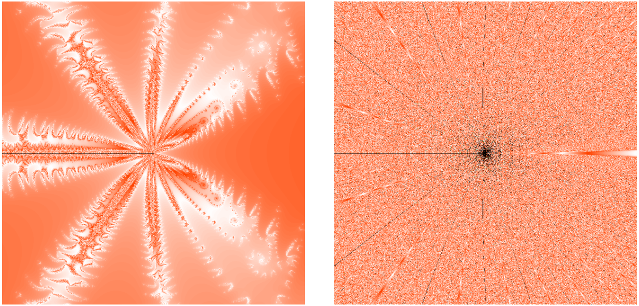
$$\mathcal{H}(x, y) = (y + x^2 + t, x).$$

Let $b \in \mathbb{C}$, and assume that $b \notin \overline{\mathbb{Z}}$. Then

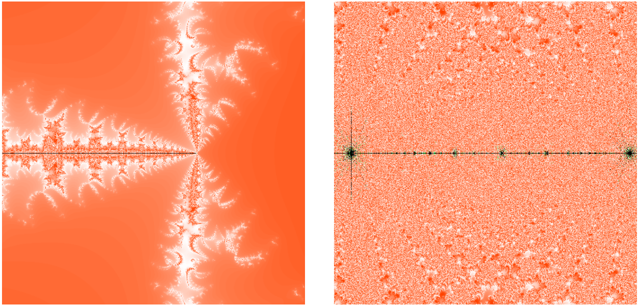
$$\Sigma((0, b)) := \{t \in \mathbb{A}^1(\overline{K}) \mid (0, b) \text{ is periodic under } H_t\} = \emptyset.$$

Example

We have $\Sigma((0, 1/2)) = \emptyset$, while $|\Sigma((a, -a))| = \infty$ for any $a \in \mathbb{C}$.



$\mathcal{P} = (0,0)$ near $t = 0$. (Right) enlarged 10^3 times. $|\Sigma(\mathcal{P})| = \infty$



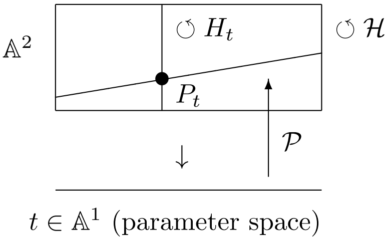
$\mathcal{P} = (0,1/2)$ near $t = -0.1$. (Right) enlarged 10^3 times. $\Sigma(\mathcal{P}) = \emptyset$

Unlikely intersection

K: a number field

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
$\mid P_t$ is periodic under $H_t\}$

We have instances
that $\Sigma(\mathcal{P})$ is infinite.



Unlikely intersection

K: a number field

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
$\quad \mid P_t$ is periodic under $H_t\}$

We have instances
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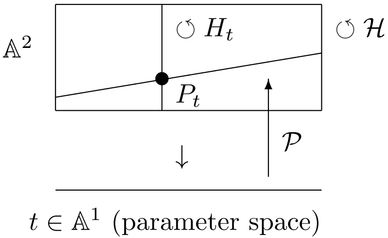

Let $\mathcal{Q} : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1$ be another section with $\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{Q}_\eta) \neq 0$.

Unlikely intersection

K: a number field

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
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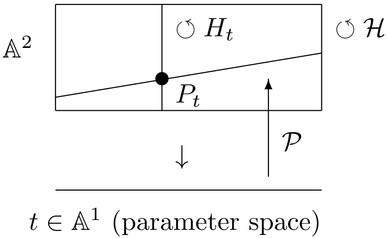

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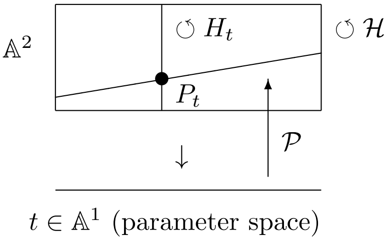

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We would like to consider when there are infinitely many parameter values t such that both P_t and Q_t are periodic under H_t.

Recall that we have shown

- $h_{\mathcal{P}}(t) := \widetilde{h}_{H_t}(P_t)/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta)$ is the restriction of the semipositive adelically metrized line bundle $\overline{\mathcal{L}_{\mathcal{P}}} := (\mathcal{O}_{\mathbb{P}^1}(1), \{\| \cdot \|\}_v)$.
- $\Sigma(P) = \{t \in \mathbb{A}^1(\overline{K}) \mid h_{\mathcal{P}}(t) = 0\}$.

Unlikely intersection (continued)

$h_{\mathcal{P}} \colon \mathbb{A}^1(\overline{K}) \to \mathbb{R}_{\geq 0}$ defined by $h_{\mathcal{P}}(t) := \widetilde{h}_{H_t}(P_t)/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta)$

We have $h_{\mathcal{P}} = c\sum_{v \in M_K} n_v G_{\mathcal{P},v}$ with $c = 1/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) > 0$

$$\begin{aligned}\Sigma(\mathcal{P}) &:= \{t \in \mathbb{A}^1(\overline{K}) \mid P_t \text{ is periodic under } H_t\} \\ &= \{t \in \mathbb{A}^1(\overline{K}) \mid h_{\mathcal{P}}(t) = 0\}\end{aligned}$$

Theorem (Hsia–K)

Assume that $\Sigma(\mathcal{P})$ and $\Sigma(\mathcal{Q})$ are infinite.

Then the following are equivalent.

1. $\Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$ *is infinite. Namely, there are infinitely many periodic values t such that P_t and Q_t are both periodic under H_t.*
2. $\Sigma(\mathcal{P}) = \Sigma(\mathcal{Q})$
3. $G_{\mathcal{P},v} = G_{\mathcal{Q},v}$ *for all $v \in M_K$.*
4. $h_{\mathcal{P}} = h_{\mathcal{Q}}$.

Unlikely intersection (continued)

Proof uses Yuan's equidistribution theorem. (Or, as the parameter space is 1-dimensional, so we can also use equidistribution theorem due to Autisser, Thuillier, Chambert-Loir, Baker–Rumely, Favre–Rivera-Letlier ...).

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Indeed, suppose that $\Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$ is infinite.
Let $\{x_n\}_{n\geq 1}$ be a sequence of distinct points with $\{x_n\} \subseteq \Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$.

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Let $\{x_n\}_{n\geq 1}$ be a sequence of distinct points with $\{x_n\} \subseteq \Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$.
Since $\{x_n\}_{n\geq 1}$ has height 0 with respect to $h_{\mathcal{P}} = h_{\overline{\mathcal{L}_{\mathcal{P}}}}$, the equidistribution theorem implies that, as $n \to \infty$, the Galois orbit of x_n will be equidistributed on $\mathbb{P}^1(\mathbb{K}_v)$ with respect to the measure $c_1(\overline{\mathcal{L}_{\mathcal{P}}})_v$ for any $v \in M_K$.

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It follows that $G_{\mathcal{P},v} = G_{\mathcal{Q},v}$ for all $v \in M_K$, and $h_{\mathcal{P}} = h_{\mathcal{Q}}$. Then $\Sigma(\mathcal{P}) = \Sigma(\mathcal{Q})$.

Unlikely intersection (continued)

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Question

Suppose that $h_{\mathcal{P}} = h_{\mathcal{Q}}$. Then does there exist an automorphism $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ over $\mathbb{A}^1$ and a positive integer $m \geq 1$ with $\sigma^{-1} \circ \mathcal{H}^m \circ \sigma = \mathcal{H}^m$ or $\sigma^{-1} \circ \mathcal{H}^m \circ \sigma = \mathcal{H}^{-m}$ such that $\mathcal{Q} = \mathcal{H}^n(\sigma(\mathcal{P}))$ for some $n \geq 1$?